

# The XXZ spin-1/2 chain: an integrable approach

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## 1 Introduction

## 2 A detour through the six vertex model

### 2.1 The transfer matrix formalism for vertex models

Vertex models constitute a large class of models of two-dimensional statistical mechanics. They can be defined on any graph, although here we shall only discuss the specific example of a two-dimensional square lattice with  $L \times M$  vertices, aka sites. In such a setting, one defines the model as follows. Each edge is endowed with a discrete degree of freedom  $\alpha$  that takes values in integers. We shall restrict to a finite number of degrees of freedom, *i.e.*  $\alpha \in \llbracket 1; n \rrbracket$ . An edge configuration around a given vertex may be represented as given in Fig. 1, and then a possible configurations of the full model can be represented as in Fig. 2. It consists of the data

$$C = \{ \alpha_{ij}, \epsilon_{ij} \in \llbracket 1; n \rrbracket \text{ with } (i, j) \in \llbracket 0; M \rrbracket \times \llbracket 0; L \rrbracket \} \quad (2.1)$$

#### boundary conditions ?

As is customary in statistical mechanics, the configurations of edges around a local vertex are random variables and as such, any given configuration of edges  $\{\alpha_{ij}, \epsilon_{ij}\}$  of the full  $L \times M$  grid is assigned a certain probability. This probability may be build starting from the local weights associated with each vertex in the following manner. First of all, focusing on a local vertex as depicted in Fig. 1, a configuration  $\alpha, \epsilon, \alpha', \epsilon'$  read by going clockwise starting from the bottom edge carries a weight

$$R_{\epsilon}^{\epsilon'} \alpha^{\alpha'} \geq 0. \quad (2.2)$$

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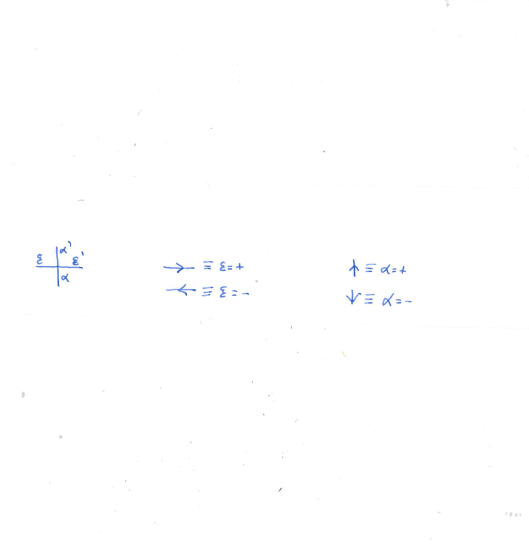


Figure 1: Conventions for denoting the weights.

In order to connect this formalism with the one typical to the statistical mechanics' parameterisation of weights, one may reparameterise the above weight as

$$\mathbf{R}_{\epsilon}^{\epsilon' \alpha'} = e^{-\frac{1}{T}v(\epsilon, \epsilon', \alpha, \alpha')} . \quad (2.3)$$

In such a setting,  $T$  is the temperature while  $v$  is some  $T$ -independent function on the possible configurations of edges attached to a given vertex, viz.  $\llbracket 1; n \rrbracket^4$  in our setting. We stress that the parametrisation (2.2) is more general than (2.3) as it allows for a more general dependence on the temperature.

The form of the weight associated with a generic local vertex being settled, when considering the configurations of all vertices

$$(i, j) , \quad i \in \llbracket 1; L \rrbracket \quad \text{and} \quad j \in \llbracket 1; M \rrbracket \quad (2.4)$$

one may allow for a non-uniformness of the weights, *i.e.* a dependence of the weights in respect to the vertex position  $(i, j)$  whose surrounding configuration of edges it weights. Going back to Fig. 2, the vertex  $(i, j)$  located at the intersection of line  $i$  and column  $j$  will have edge configurations (going clockwise starting from the bottom edge)  $\alpha_{i-1, j}, \epsilon_{i, j-1}, \alpha_{i, j}, \epsilon_{i, j}$ . One then agrees to associate to this configuration the weight

$$\left[ \mathbf{R}^{(i, j)} \right]_{\epsilon_{i, j-1} \alpha_{i-1, j}}^{\epsilon_{i, j} \alpha_{i, j}} . \quad (2.5)$$

Finally, the probability of observing the configuration of vertices  $\{\alpha_{ij}, \epsilon_{ij}\}$  of the full  $L \times M$  grid is obtained by taking the properly normalised product of all local weights, *i.e.*

$$\mathbb{P}[\{\alpha_{ij}, \epsilon_{ij}\}] = \frac{1}{\mathcal{Z}_{gen;bc}} \prod_{i=1}^M \prod_{j=1}^L \left\{ \left[ \mathbf{R}^{(i, j)} \right]_{\epsilon_{i, j-1} \alpha_{i-1, j}}^{\epsilon_{i, j} \alpha_{i, j}} \right\} \quad (2.6)$$

where  $\mathcal{Z}_{gen;bc}$  is a normalisation factor, called the partition function. It is chosen in such a way that the *lhs* above does correspond to a probability density.

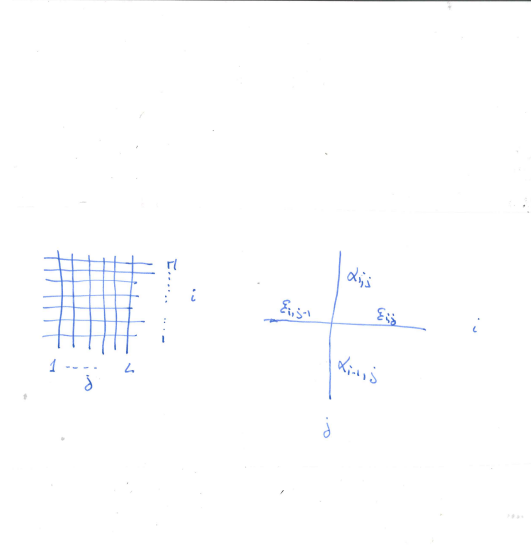


Figure 2: Local parameterisation of edge labels.

So far, we did not treat in any specific way the degrees of freedom of the "outer" edges of the  $L \times M$  grid, *viz.* those that are only connected to *one* vertex of the grid, as opposed to the "bulk" edges which connect *two* neighbouring vertices of the grid. The outer degrees of freedom which correspond to the labels:

$$\epsilon_{i,0}, \epsilon_{i,L} \text{ for } i \in \llbracket 1; M \rrbracket \quad \text{and} \quad \alpha_{0,j}, \alpha_{M,j} \text{ for } j \in \llbracket 1; L \rrbracket . \quad (2.7)$$

While one allows the "inner" edges to take all values in  $\llbracket 1; n \rrbracket$  as one runs through all the possible edge configurations of the grid, one usually adds some constraints on the values taken by the outer edges. This corresponds to specifying the boundary conditions. There are various kinds of possible boundary conditions. One may consider so-called free boundary conditions. These correspond to simply allowing any value for the outer edges, *viz.* treating them as "bulk" degrees of freedom:

$$\epsilon_{i,0}, \epsilon_{i,L}, \alpha_{0,j}, \alpha_{M,j} \in \llbracket 1; n \rrbracket \text{ for } i \in \llbracket 1; M \rrbracket \text{ and } j \in \llbracket 1; L \rrbracket . \quad (2.8)$$

One may also consider fixed boundary conditions,

$$\epsilon_{i,0} = u_i^{(\leftarrow)}, \quad \epsilon_{i,L} = u_i^{(\rightarrow)}, \quad \alpha_{0,j} = u_j^{(\downarrow)}, \quad \alpha_{M,j} = u_j^{(\uparrow)}, \quad (2.9)$$

for some *fixed* sequences  $u_i^{(\leftarrow)}, u_i^{(\rightarrow)}, u_j^{(\downarrow)}, u_j^{(\uparrow)} \in \llbracket 1; n \rrbracket$ . Finally, another type of interesting boundary conditions pertains to the periodic ones:

$$\epsilon_{i,0} = \epsilon_{i,L} \text{ for } i \in \llbracket 1; M \rrbracket \quad \text{and} \quad \alpha_{0,j} = \alpha_{M,j} \in \llbracket 1; n \rrbracket \text{ for } j \in \llbracket 1; L \rrbracket . \quad (2.10)$$

Morally speaking this last boundary condition may be understood as issuing from the topology of the ambient space on which the grid is drawn, the torus in that case.

For the time-being, we shall focus on the periodic boundary conditions. At a later stage when we'll have already introduced enough formalism, we will discuss a particular case of fixed boundary conditions, called domain-wall boundary conditions.

It will appear convenient, for later purposes, to gather the weights associated with a given vertex into an operator  $R$  acting on a tensor product space  $\mathfrak{h} \otimes \mathfrak{h}$  with  $\mathfrak{h} = \mathbb{C}^n$ . We shall endow  $\mathfrak{h}$  with the basis  $\{e^1, \dots, e^n\}$ . Hence,  $\mathfrak{h} \otimes \mathfrak{h}$  has basis  $\{e^1 \otimes e^1, e^1 \otimes e^2, \dots, e^1 \otimes e^n, e^2 \otimes e^1, \dots, e^n \otimes e^n\}$ . Then, any  $R \in \mathcal{L}(\mathfrak{h} \otimes \mathfrak{h})$  may be characterised by its matrix elements

$$R_{\epsilon \alpha}^{\epsilon' \alpha'} \quad \text{so that} \quad R \cdot e^{\epsilon'} \otimes e^{\alpha'} = \sum_{\{\epsilon, \alpha\}} R_{\epsilon \alpha}^{\epsilon' \alpha'} e^{\epsilon} \otimes e^{\alpha}. \quad (2.11)$$

This entails the rule for matrix products

$$(AB)_{\epsilon \alpha}^{\epsilon' \alpha'} = \sum_{\{\tau, \varrho\}} A_{\epsilon \alpha}^{\tau \varrho} B_{\tau \varrho}^{\epsilon' \alpha'} \quad (2.12)$$

To make the best of this situation in the case of the  $L \times M$  grid of vertices as in Fig. 2, with each line  $i$  we associate a Hilbert space  $\mathfrak{h}_{a_i} = \mathbb{C}^n$  and with each column  $j$  a Hilbert space  $\mathfrak{h}_j = \mathbb{C}^n$ . The spaces  $\mathfrak{h}_{a_i}$  or  $\mathfrak{h}_j$  are called local. Then  $R^{(i,j)}$  may be embedded into  $\mathcal{L}(\mathfrak{h}_{a_i} \otimes \mathfrak{h}_q)$  with  $\mathfrak{h}_q = \bigotimes_{j=1}^L \mathfrak{h}_j$  as  $R_{a_i j}^{(i,j)}$ . This notation means that

$$R_{a_i 1}^{(i,j)} = R^{(i,j)} \otimes \underbrace{\text{id} \otimes \dots \otimes \text{id}}_{L-1 \text{ times}} \quad \text{and} \quad R_{a_i j}^{(i,j)} = P_{1j} R_{a_i 1}^{(i,j)} P_{1j}. \quad (2.13)$$

There,  $P_{1j}$  is the permutation operator between spaces  $\mathfrak{h}_1$  and  $\mathfrak{h}_j$ :

$$P_{1j}(e_{a_i} \otimes v_1 \otimes \dots \otimes v_L) = e_{a_i} \otimes v_j \otimes v_2 \otimes \dots \otimes v_{j-1} \otimes v_1 \otimes v_{j+1} \otimes \dots \otimes v_L. \quad (2.14)$$

Within the introduced formalism, and under periodic boundary conditions (2.10), we are now in position to evaluate, in a simpler way, the associated normalising factor, *aka* the partition function of the model subject to periodic boundary conditions:

$$\mathcal{Z}_{\text{gen;per}} = \sum_{C \in \mathcal{E}_{\text{per}}} \prod_{i=1}^M \prod_{j=1}^L \left\{ [R^{(i,j)}]_{\epsilon_{i,j-1} \alpha_{i-1,j}}^{\epsilon_{i,j} \alpha_{i,j}} \right\}. \quad (2.15)$$

Above, the summation runs through all the possible configurations  $C = \{\epsilon_{i,j}, \alpha_{i,j}\}$  of the grid's edges:

$$\mathcal{E}_{\text{per}} = \left\{ \alpha_{i,j}, \epsilon_{i,j} \in \llbracket 1; n \rrbracket \quad \text{for} \quad i \in \llbracket 1; M \rrbracket \quad \text{and} \quad j \in \llbracket 1; L \rrbracket \quad \text{and} \quad \epsilon_{i,0} = \epsilon_{i,L}, \quad \alpha_{0,j} = \alpha_{M,j} \right\} \quad (2.16)$$

The purely combinatorial expression for  $\mathcal{Z}_{\text{gen;per}}$  given in (2.15) may be recast in an elegant way by means of the operator notations we have just established. Indeed, one may reorganise the sums as

$$\begin{aligned} \mathcal{Z}_{\text{gen;per}} &= \sum_{\{\alpha_{i,j}\}_{i=1, j=1}^{M,L}} \left\{ \sum_{\{\epsilon_{1,j}\}_{j=1}^M} [R^{(1,1)}]_{\epsilon_{1,0} \alpha_{0,1}}^{\epsilon_{1,1} \alpha_{1,1}} [R^{(1,2)}]_{\epsilon_{1,1} \alpha_{0,2}}^{\epsilon_{1,2} \alpha_{1,2}} \cdot [R^{(1,L)}]_{\epsilon_{1,L-1} \alpha_{0,L}}^{\epsilon_{1,L} \alpha_{1,L}} \right\} \\ &\times \left\{ \sum_{\{\alpha_{2,j}\}_{j=1}^M} [R^{(2,1)}]_{\epsilon_{2,0} \alpha_{1,1}}^{\epsilon_{2,1} \alpha_{2,1}} [R^{(2,2)}]_{\epsilon_{2,1} \alpha_{1,2}}^{\epsilon_{2,2} \alpha_{2,2}} \cdot [R^{(2,L)}]_{\epsilon_{2,L-1} \alpha_{1,L}}^{\epsilon_{2,L} \alpha_{2,L}} \right\} \\ &\times \left\{ \sum_{\{\alpha_{M,j}\}_{j=1}^M} [R^{(M,1)}]_{\epsilon_{M,0} \alpha_{M-1,1}}^{\epsilon_{M,1} \alpha_{M,1}} [R^{(M,2)}]_{\epsilon_{M,1} \alpha_{M-1,2}}^{\epsilon_{M,2} \alpha_{M,2}} \cdot [R^{(M,L)}]_{\epsilon_{M,L-1} \alpha_{M-1,L}}^{\epsilon_{M,L} \alpha_{M,L}} \right\}. \quad (2.17) \end{aligned}$$

There, for fixed  $i$ , it is understood that

$$\sum_{\{\epsilon_{i,j}\}_{j=1}^L} = \sum_{\epsilon_{i,1}=1}^n \cdots \sum_{\epsilon_{i,L}=1}^n . \quad (2.18)$$

Now, for fixed  $i \in \llbracket 1 ; M \rrbracket$ , each sum may be recast as

$$\begin{aligned} \sum_{\{\epsilon_{i,j}\}} \left[ \mathbf{R}^{(i,1)} \right]_{\epsilon_{i,0}\alpha_{i-1,1}}^{\epsilon_{i,1} \alpha_{i,1}} \left[ \mathbf{R}^{(i,2)} \right]_{\epsilon_{i,1}\alpha_{i-1,2}}^{\epsilon_{i,2} \alpha_{i,2}} \cdots \left[ \mathbf{R}^{(i,L)} \right]_{\epsilon_{i,L-1}\alpha_{i-1,L}}^{\epsilon_{i,L} \alpha_{i,L}} \\ = \sum_{\{\epsilon_{i,L}\}} \left[ \mathbf{R}^{(i,1)} \mathbf{R}^{(i,2)} \cdots \mathbf{R}^{(i,L)} \right]_{\epsilon_{i,0}\alpha_{i-1,1}, \dots, \alpha_{i-1,L}}^{\epsilon_{i,L} \alpha_{i,1} \dots \alpha_{i,L}} = \text{tr}_{\mathfrak{h}_{a_i}} \left[ \mathbf{T}_{a_i; \mathfrak{q}}^{(i)} \right]_{\alpha_{i-1,1}, \dots, \alpha_{i-1,L}}^{\alpha_{i,1} \dots \alpha_{i,L}} . \end{aligned} \quad (2.19)$$

Above, we have introduced the monodromy matrix on  $\mathfrak{h}_{a_i} \otimes \mathfrak{h}_{\mathfrak{q}}$

$$\mathbf{T}_{a_i; \mathfrak{q}}^{(i)} = \mathbf{R}_{a_i 1}^{(i,1)} \cdots \mathbf{R}_{a_i L}^{(i,L)} , \quad (2.20)$$

and computed its partial trace over  $\mathfrak{h}_{a_i}$ . The resulting quantity  $\text{tr}_{\mathfrak{h}_{a_i}} \left[ \mathbf{T}_{a_i; \mathfrak{q}}^{(i)} \right]$  is called a transfer matrix and is an operator on  $\mathfrak{h}_{\mathfrak{q}}$

This recasts the partition function as

$$\begin{aligned} \mathcal{Z}_{\text{gen;per}} &= \sum_{\{\alpha_{i,j}\}_{i=1, j=1}^{L,M}} \text{tr}_{\mathfrak{h}_{a_1}} \left[ \mathbf{T}_{a_1; \mathfrak{q}}^{(1)} \right]_{\alpha_{0,1}, \dots, \alpha_{0,L}}^{\alpha_{1,1}, \dots, \alpha_{1,L}} \cdots \text{tr}_{\mathfrak{h}_{a_2}} \left[ \mathbf{T}_{a_2; \mathfrak{q}}^{(2)} \right]_{\alpha_{1,1}, \dots, \alpha_{1,L}}^{\alpha_{2,1}, \dots, \alpha_{2,L}} \cdots \text{tr}_{\mathfrak{h}_{a_M}} \left[ \mathbf{T}_{a_M; \mathfrak{q}}^{(M)} \right]_{\alpha_{M-1,1}, \dots, \alpha_{M-1,L}}^{\alpha_{M,1}, \dots, \alpha_{M,L}} \\ &= \sum_{\{\alpha_{0,1}, \dots, \alpha_{0,M}\}} \left[ \text{tr}_{\mathfrak{h}_{a_1}} \left[ \mathbf{T}_{a_1; \mathfrak{q}}^{(1)} \right] \cdots \text{tr}_{\mathfrak{h}_{a_M}} \left[ \mathbf{T}_{a_M; \mathfrak{q}}^{(M)} \right] \right]_{\alpha_{0,1}, \dots, \alpha_{0,M}}^{\alpha_{M,1}, \dots, \alpha_{M,M}} = \text{tr}_{\mathfrak{h}_{\mathfrak{q}}} \left[ \text{tr}_{\mathfrak{h}_{a_1}} \left[ \mathbf{T}_{a_1; \mathfrak{q}}^{(1)} \right] \cdots \text{tr}_{\mathfrak{h}_{a_M}} \left[ \mathbf{T}_{a_M; \mathfrak{q}}^{(M)} \right] \right] . \end{aligned} \quad (2.21)$$

So far, the formalism allows one to boil down the summation problem into a linear algebra problem.

Evaluating this normalisation factor, in the thermodynamic limit constitutes the first physically interesting quantities associated with such a model. The associated quantity is called the per-site free energy and is defined by the below limit<sup>†</sup>

$$\mathfrak{f}_{\text{gen;per}} = - \lim_{M, L \rightarrow \infty} \frac{1}{ML} \ln \mathcal{Z}_{\text{gen;per}} . \quad (2.22)$$

Several questions immediately arise:

- i) What is the meaning of the limit, *viz.* how do  $M, L$  should approach to infinity?
- ii) For which class of weights is the limit well-defined?
- iii) When defined, is it possible to say more about the limit, for instance how it behaves as a function of the additional parameters that may enter in the expression for the weights, such as the temperature as in (2.3)?

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<sup>†</sup>Is is customary in statistical mechanics to define the per-site free energy with a temperature  $T$ -factor appearing in the *rhs* of (2.22). However, since we use the general parametrisation of weights -and not the Boltzmann factor related one as given in (2.3)- we find that dropping it would be more natural in our context.

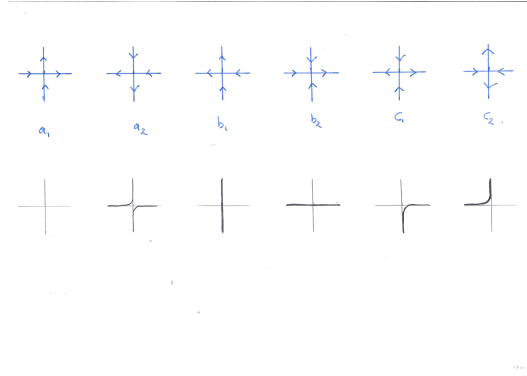


Figure 3: Allowed local configurations in the six-vertex model.

**check with Lieb** The formalism of rigorous statistical mechanics [5] provides a general background for taking such thermodynamic, *viz.* when the size of the discrete lattice approaches infinity.

One may wonder if alternative ways of computing the limit, for instance as a successive limit

$$- \lim_{M \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{1}{ML} \ln \mathcal{Z}_{\text{gen;per}} \quad \text{or} \quad - \lim_{L \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{1}{ML} \ln \mathcal{Z}_{\text{gen;per}} \quad (2.23)$$

also exist and whether these limits provide one with the same quantity  $f_{\text{gen;per}}$ . We shall establish such a statement in a later part of these lectures.

However, even when one is in some nice setting allowing one to deal with points *i*) and *ii*), answering question *iii*) turn out to be notoriously hard.

In the following, we will not try to address these questions in full generality, but shall rather focus in answering these -and many other- in the case of a very specific model: the six-vertex model. As we shall see, this model is closely connected to the XXZ spin-1/2 Hamiltonian in 1 dimensions mentioned in the introduction. The reason is that this specific model possesses a hidden algebraic structure which allows one to deduce the answer to question *iii*) and many other related ones.

## 2.2 The six-vertex model

The six-vertex model is a very specific case of a vertex model on an  $L \times M$  grid, where each vertex has two possible configurations, *viz.*  $n = 2$ . As a consequence, these can also be labelled by  $\{\pm\}$  or, equivalently, by incoming or out-going arrows, as depicted in Fig. 1. There are, in total, six possible configurations of arrows around a vertex as depicted in Fig. 3. All other configurations are not allowed, *viz.* correspond to a zero weight.

One may denote the weights associated to the allowed configurations as  $a_1, a_2, b_1, b_2, c_1, c_2$ , *c.f.* Fig. 3. One may solve the model [1], *i.e.* access to explicit expressions for numerous physically interesting observables, for any choices of  $a_1, a_2, b_1, b_2, c_1, c_2$ . However, for the sake of simplifying the exposition, we shall henceforth limit ourselves to the case

$$a_1 = a_2 = a, \quad b_1 = b_2 = b \quad \text{and} \quad c_1 = c_2 = c, \quad (2.24)$$

which already exhibits very rich physical properties.

Let  $\mathfrak{h} \simeq \mathbb{C}^2$  be endowed with the basis  $\{e^+, e^-\}$ . Then,  $\mathfrak{h} \otimes \mathfrak{h}$  has basis  $\{e^+ \otimes e^+, e^+ \otimes e^-, e^- \otimes e^+, e^- \otimes e^-\}$

The weights of the six-vertex model on which we focus may be gathered in a  $4 \times 4$  matrix  $R$ :

$$R = \begin{pmatrix} R_{++}^{++} & 0 & 0 & 0 \\ 0 & R_{+-}^{+-} & R_{+-}^{+} & 0 \\ 0 & R_{-+}^{+-} & R_{-+}^{+} & 0 \\ 0 & 0 & 0 & R_{--}^{--} \end{pmatrix} \quad \text{with} \quad \begin{cases} R_{++}^{++} = R_{--}^{--} = a \\ R_{+-}^{+-} = R_{-+}^{+} = b \\ R_{-+}^{+} = R_{+-}^{+-} = c \end{cases} . \quad (2.25)$$

In fact, one may conveniently parameterise these in terms of three parameters  $\gamma, \lambda$  and  $\eta$  as

$$R(\lambda) = \gamma \begin{pmatrix} \sinh(\lambda + \eta) & 0 & 0 & 0 \\ 0 & \sinh(\lambda) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(\lambda) & 0 \\ 0 & 0 & 0 & \sinh(\lambda + \eta) \end{pmatrix} . \quad (2.26)$$

This is the so-called six-vertex  $R$ -matrix in polynomial normalisation.

While for generic complex values of  $\gamma, \lambda$  and  $\eta$  the weights are genuinely complex valued, however there are several possible choices which lead to positive weights.

First of all, the choices  $\lambda, \eta, \gamma \in \mathbb{R}^+$  and  $\lambda, \eta, \gamma \in \mathbb{R}^-$  yield that the three weights

$$[R]_{++}^{++} = \gamma \sinh(\lambda + \eta), \quad [R]_{+-}^{+-} = \gamma \sinh(\lambda) \quad \text{and} \quad [R]_{-+}^{+-} = \gamma \sinh(\eta), \quad (2.27)$$

are all strictly positive. Further, the choice

$$\eta = i(\pi - \zeta), \quad \gamma = \frac{-ir}{\sin(\zeta/2)}, \quad \lambda = i\frac{\theta\zeta}{\pi}, \quad (2.28)$$

with  $\zeta, \theta \in ]0; \pi[$  and  $r \in \mathbb{R}^+$ , yields that the three weights

$$[R]_{++}^{++} = r \frac{\sin[(1 - \theta/\pi)\zeta]}{\sin[\zeta/2]}, \quad [R]_{+-}^{+-} = r \frac{\sin[\theta\zeta/\pi]}{\sin[\zeta/2]} \quad \text{and} \quad [R]_{-+}^{+-} = 2r \cos[\zeta/2], \quad (2.29)$$

are also all strictly positive.

The form of the local weight being settled, when passing to the weights for the full  $L \times M$  grid, as earlier on, one may also allow the weights to depend in a certain way on the vertex position  $(i, j)$  by taking  $R^{(i,j)} = R(\lambda_i - \xi_j)$  in which  $R(\lambda)$  is the six-vertex  $R$ -matrix in polynomial normalisation given above (2.26). Here, in principle,  $\lambda_i, \xi_k$  are some free parameters. However, one should take these of a specific type for all local weights to be real-valued and positive. Namely, either one takes  $\gamma, \eta \in \mathbb{R}^+$  and  $\lambda_j - \xi_k \in \mathbb{R}^+$ , or  $\gamma, \eta \in \mathbb{R}^-$  and  $\lambda_j - \xi_k \in \mathbb{R}^-$  or

$$\eta = i(\pi - \zeta), \quad \gamma = \frac{-ir}{\sin(\zeta/2)}, \quad \lambda_k = -i\frac{\zeta}{2} + i\frac{\theta_k\zeta}{\pi} \quad \text{and} \quad \xi_k = -i\frac{\zeta}{2} + i\frac{\sigma_k\zeta}{\pi}, \quad (2.30)$$

where  $\zeta \in ]0; \pi[$ ,  $\lambda_j - \xi_k \in ]0; \pi[$  and  $r \in \mathbb{R}^+$ .

By introducing the monodromy matrix on  $\mathfrak{h}_{a_i} \otimes \mathfrak{h}_q$

$$T_{a_i; q}(\lambda | \xi_L) = R_{a_i 1}(\lambda - \xi_1) \cdots R_{a_i L}(\lambda - \xi_L) \quad \text{with} \quad \xi_L = (\xi_1, \dots, \xi_L), \quad (2.31)$$

and the associated transfer matrix

$$t(\lambda | \xi_L) = \text{tr}_{\mathfrak{h}_{a_i}} [T_{a_i; q}(\lambda | \xi_L)], \quad (2.32)$$

one may identify the partition function  $\mathcal{Z}_{6V; \text{per}}$  subordinate to such vertex dependent weights as the below trace

$$\mathcal{Z}_{6V; \text{per}} = \text{tr}_{\mathfrak{h}_q} [t(\lambda_1 | \xi_L) \cdots t(\lambda_M | \xi_L)]. \quad (2.33)$$

We are now going to investigate in greater details the algebraic properties of the various building blocks of the partition function.

### 2.3 The algebraic structure at the root of the six-vertex model

The main building block of the operators -monodromy and transfer matrices- which provide a simple algebraic expression for  $\mathcal{Z}_{6V;\text{per}}$  is the so-called six-vertex  $R$ -matrix given in (2.26). One may check that this matrix satisfies an algebraically very nice equation called the Yang-Baxter equation

$$R_{12}(\lambda - \mu) R_{13}(\lambda - \nu) R_{23}(\mu - \nu) = R_{23}(\mu - \nu) R_{13}(\lambda - \nu) R_{12}(\lambda - \mu) . \quad (2.34)$$

We shall not establish the validity of this equation here. This can be done through direct calculations based on elementary algebra of trigonometric functions. The equation may also be checked directly as follows. First, one observes that both,  $lhs$  and  $rhs$  are hyperbolic polynomials of degree 2, *viz.* is of the form  $\sum_{s=-2}^2 e^{s\lambda} C_s$  for some coefficients  $C_s$ . It is thus enough to check the validity of the equation at 6 points, *e.g.*  $\lambda \rightarrow \pm\infty, \lambda = \mu, \lambda = \nu$  and  $\lambda = \mu - \eta, \lambda = \nu - \eta$ .

The Yang-Baxter equation is at the root of the integrability of the 6 vertex model. In fact, when properly generalised - this aspect will not be tackled in the present series of lectures- this equation is at the root on theory of quantum integrable systems. The equation originally appeared in quite different contexts, in the works of McGuire [4], Yang [6] in 1967 and Baxter [2] in 1972. However, it was only in 1979 that it was raised to full glory by Faddeev, Sklyanin and Takhtadjan [3] who showed how to use the algebra generated by this equation for obtaining the exact solution of an integrable model. We shall develop this formalism in this subsection.

First of all, we introduce new objects called Lax matrices

$$L_{a;n}(\lambda) = \gamma \left( \begin{array}{cc} \sinh(\lambda + \frac{\eta}{2}[1 + \sigma_n^z]) & \sinh(\eta) \cdot \sigma_n^- \\ \sinh(\eta) \cdot \sigma_n^+ & \sinh(\lambda + \frac{\eta}{2}[1 - \sigma_n^z]) \end{array} \right)_{[a_i]} = R_{a;n}(\lambda) . \quad (2.35)$$

Here  $\sigma^z, \sigma^\pm$  are the Pauli matrices

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.36)$$

while  $\sigma_n^\alpha$  stand for their embeddings as operators on  $\mathfrak{h}_q = \bigotimes_{a=1}^L \mathfrak{h}_a$ :

$$\sigma_n^\alpha = \underbrace{\text{id} \otimes \cdots \otimes \text{id}}_{n-1 \text{ times}} \otimes \sigma^\alpha \otimes \underbrace{\text{id} \otimes \cdots \otimes \text{id}}_{L-n \text{ times}} . \quad (2.37)$$

Further, the index  $[a_i]$  in the matrix indicates the space  $\mathfrak{h}_{a_i}$ - in respect to which one writes the matrix representation. Note that the entries of this matrix act as non-trivial operators on  $\mathfrak{h}_n$ . Obviously, by construction,  $L$  satisfies the Yang-Baxter equation:

$$R_{ab}(\lambda - \mu) L_{an}(\lambda - \xi_n) L_{bn}(\mu - \xi_n) = L_{bn}(\mu - \xi_n) L_{an}(\lambda - \xi_n) R_{ab}(\lambda - \mu) . \quad (2.38)$$

With these new notation at hand, the monodromy matrix may be written as

$$T_{a_i;q}(\lambda \mid \xi_L) = L_{a_i 1}(\lambda - \xi_1) \cdots L_{a_i L}(\lambda - \xi_L) . \quad (2.39)$$

The first impressive result which follows from the algebraic structure at the root of the Yang-Baxter equation is that  $T_{a_i;q}$  also satisfies the Yang-Baxter equation.

**Proposition 2.1.** *It holds*

$$R_{ab}(\lambda - \mu) T_{a;q}(\lambda \mid \xi_L) T_{b;q}(\mu \mid \xi_L) = T_{b;q}(\mu \mid \xi_L) T_{a;q}(\lambda \mid \xi_L) R_{ab}(\lambda - \mu) . \quad (2.40)$$

*Proof*—

By using the ultralocal structure of the L matrices, *viz.* that  $L_{an}(\lambda - \xi_n)$  only involves a non trivial action on the spaces  $\mathfrak{h}_a$  and  $\mathfrak{h}_n$  and thus commutes with all operators acting non-trivially on other spaces that  $\mathfrak{h}_a$  and  $\mathfrak{h}_n$ , one may re-organise the *lhs* of (2.40) in the form

$$\begin{aligned} R_{ab}(\lambda - \mu) T_{a;q}(\lambda | \xi_L) T_{b;q}(\mu | \xi_L) &= R_{ab}(\lambda - \mu) L_{a1}(\lambda - \xi_1) L_{b1}(\mu - \xi_1) L_{a2}(\lambda - \xi_2) L_{b2}(\mu - \xi_2) \\ &\quad \cdots L_{aL}(\lambda - \xi_L) L_{bL}(\mu - \xi_L) . \end{aligned} \quad (2.41)$$

Now, one may use the Yang-Baxter equation satisfied by L to successively exchange the "a" and "b" Lax matrices

$$\begin{aligned} R_{ab}(\lambda - \mu) T_{a;q}(\lambda | \xi_L) T_{b;q}(\mu | \xi_L) &= L_{b1}(\mu - \xi_1) L_{a1}(\lambda - \xi_1) R_{ab}(\lambda - \mu) L_{a2}(\lambda - \xi_2) L_{b2}(\mu - \xi_2) \\ \cdots L_{aL}(\lambda - \xi_L) L_{bL}(\mu - \xi_L) &= L_{b1}(\mu - \xi_1) L_{a1}(\lambda - \xi_1) \cdots L_{bj-1}(\mu - \xi_{j-1}) L_{aj-1}(\lambda - \xi_{j-1}) R_{ab}(\lambda - \mu) \\ &\quad \times L_{aj}(\lambda - \xi_j) L_{bj}(\mu - \xi_j) \cdots L_{aL}(\lambda - \xi_L) L_{bL}(\mu - \xi_L) = T_{b;q}(\mu | \xi_L) T_{a;q}(\lambda | \xi_L) R_{ab}(\lambda - \mu) . \end{aligned} \quad (2.42)$$

This entails the claim. ■

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