The XXZ spin-$1/2$ chain: an integrable approach

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2.1 The transfer matrix formalism for vertex models

Vertex models constitute a large class of models of two-dimensional statistical mechanics. They can be defined on any graph, although here we shall only discuss the specific example of a two-dimensional square lattice with $L \times M$ vertices, aka sites. In such a setting, one defines the model as follows. Each edge is endowed with a discrete degree of freedom α that takes values in integers. We shall restrict to a finite number of degrees of freedom, *i.e.* $\alpha \in [\![1 ; n]\!]$. An edge configuration around a given vertex may be represented as given in Fig. 1, and then a possible configurations of the full model can be represented as in Fig. 2. It consists of the data

$$C = \Big\{ \alpha_{ij}, \epsilon_{ij} \in [\![1 ; n]\!] \quad \text{with} \quad (i, j) \in [\![0 ; M]\!] \times [\![0 ; L]\!] \Big\} \tag{2.1}$$

As is customary in statistical mechanics, the configurations of edges around a local vertex are random variables and as such, any given configuration of edges $\{\alpha_{ij}, \epsilon_{ij}\}$ of the full $L \times M$ grid is assigned a certain probability. This probability may be build starting from the local weights associated with each vertex in the following manner. First

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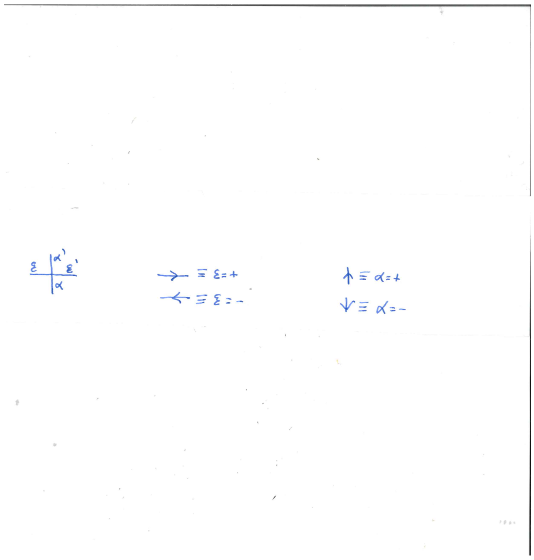

Figure 1: Conventions for denoting the weights.

of all, focusing on a local vertex as depicted in Fig. 1, a configuration $\alpha, \epsilon, \alpha', \epsilon'$ read by going clockwise starting from the bottom edge carries a weight

$$\mathrm{R}^{\epsilon' \alpha'}_{\epsilon\; \alpha} \geq 0 \; . \tag{2.2}$$

In order to connect this formalism with the one typical to the statistical mechanics' parameterisation of weights, one may reparameterise the above weight as

$$\mathrm{R}^{\epsilon' \alpha'}_{\epsilon\; \alpha} \; = \; \mathrm{e}^{-\frac{1}{T} \upsilon(\epsilon, \epsilon', \alpha, \alpha')} \; . \tag{2.3}$$

In such a setting, T is the temperature while υ is some T-independent function on the possible configurations of edges attached to a given vertex, *viz.* $[\![\, 1\, ; n \,]\!]^4$ in our setting. We stress that the parametrisation (2.2) is more general then (2.3) as it allows for a more general dependence on the temperature.

The form of the weight associated with a generic local vertex being settled, when considering the configurations of all vertices

$$(i, j) \; , \quad i \in [\![\, 1\, ; L \,]\!] \quad \text{and} \quad j \in [\![\, 1\, ; M \,]\!] \tag{2.4}$$

one may allow for a non-uniformness of the weights, *i.e.* a dependence of the weights in respect to the vertex position (i, j) whose surrounding configuration of edges it weights. Going back to Fig. 2, the vertex (i, j) located at the intersection of line i and column j will have edge configurations (going clockwise starting from the bottom edge) $\alpha_{i-1,j}, \epsilon_{i,j-1}, \alpha_{i,j}, \epsilon_{i,j}$. One then agrees to associate to this configuration the weight

$$\Big[\mathrm{R}^{(i,j)}\Big]^{\epsilon_{i,j}\; \alpha_{i,j}}_{\epsilon_{i,j-1} \alpha_{i-1,j}} \; . \tag{2.5}$$

Finally, the probability of observing the configuration of vertices $\{\alpha_{ij}, \epsilon_{ij}\}$ of the full $L \times M$ grid is obtained by taking the properly normalised product of all local weights, *i.e.*

$$\mathbb{P}\Big[\{\alpha_{ij}, \epsilon_{ij}\}\Big] \; = \; \frac{1}{\mathcal{Z}_{gen;bc}} \prod_{i=1}^{M} \prod_{j=1}^{L} \bigg\{ \Big[\mathrm{R}^{(i,j)}\Big]^{\epsilon_{i,j}\; \alpha_{i,j}}_{\epsilon_{i,j-1} \alpha_{i-1,j}} \bigg\} \tag{2.6}$$

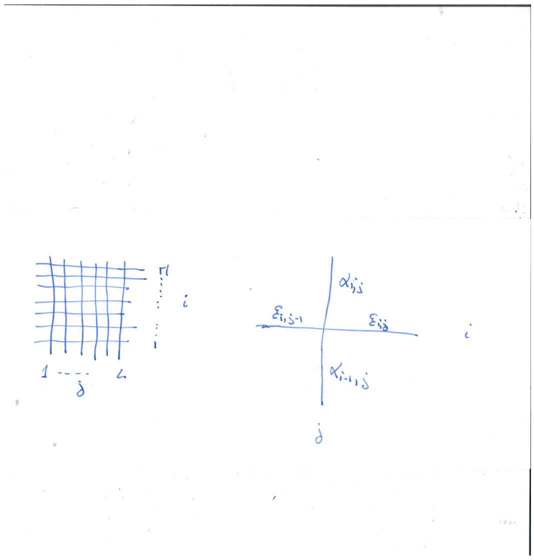

Figure 2: Local parameterisation of edge labels.

where $\mathcal{Z}_{gen;bc}$ is a normalisation factor, called the partition function. It is chosen in such a way that the *lhs* above does correspond to a probability density.

So far, we did not treat in any specific way the degrees of freedom of the "outer" edges of the $L \times M$ grid, *viz.* those that are only connected to *one* vertex of the grid, as opposed to the "bulk" edges which connect *two* neighbouring vertices of the grid. The outer degrees of freedom which correspond to the labels:

$$\epsilon_{i,0}, \epsilon_{i,L} \quad \text{for} \quad i \in [\![1 \,;\, M]\!] \qquad \text{and} \qquad \alpha_{0,j}, \alpha_{M,j} \quad \text{for} \quad j \in [\![1 \,;\, L]\!] \,. \tag{2.7}$$

While one allows the "inner" edges to take all values in $[\![1 \,;\, n]\!]$ as one runs through all the possible edge configurations of the grid, one usually adds some constraints on the values taken by the outer edges. This corresponds to specifying the boundary conditions. There are various kinds of possible boundary conditions. One may consider so-called free boundary conditions. These correspond to simply allowing any value for the outer edges, *viz.* treating them as "bulk" degrees of freedom:

$$\epsilon_{i,0}, \epsilon_{i,L}, \alpha_{0,j}, \alpha_{M,j} \in [\![1 \,;\, n]\!] \quad \text{for } i \in [\![1 \,;\, M]\!] \quad \text{and} \quad j \in [\![1 \,;\, L]\!] \,. \tag{2.8}$$

One may also consider fixed boundary conditions,

$$\epsilon_{i,0} = u_i^{(\leftarrow)} \,, \quad \epsilon_{i,L} = u_i^{(\rightarrow)} \,, \quad \alpha_{0,j} = u_j^{(\downarrow)} \,, \quad \alpha_{M,j} = u_j^{(\uparrow)} \,, \tag{2.9}$$

for some *fixed* sequences $u_i^{(\leftarrow)}, u_i^{(\rightarrow)}, u_j^{(\downarrow)}, u_j^{(\uparrow)} \in [\![1 \,;\, n]\!]$. Finally, another type of interesting boundary conditions pertains to the periodic ones:

$$\epsilon_{i,0} = \epsilon_{i,L} \quad \text{for } i \in [\![1 \,;\, M]\!] \quad \text{and} \quad \alpha_{0,j} = \alpha_{M,j} \in [\![1 \,;\, n]\!] \quad \text{for } j \in [\![1 \,;\, L]\!] \,. \tag{2.10}$$

Morally speaking this last boundary condition may be understood as issuing from the topology of the ambient space on which the grid is drawn, the torus in that case.

For the time-being, we shall focus on the periodic boundary conditions. At a later stage when we'll have already introduced enough formalism, we will discuss a particular case of fixed boundary conditions, called domain-wall boundary conditions.

It will appear convenient, for later purposes, to gather the weights associated with a given vertex into an operator R acting on a tensor product space $\mathfrak{h}\otimes\mathfrak{h}$ with $\mathfrak{h}=\mathbb{C}^n$. We shall endow $\mathfrak{h}$ with the basis $\{\boldsymbol{e}^1,\dots,\boldsymbol{e}^n\}$. Hence, $\mathfrak{h}\otimes\mathfrak{h}$ has basis $\{\boldsymbol{e}^1\otimes\boldsymbol{e}^1,\boldsymbol{e}^1\otimes\boldsymbol{e}^2,\dots,\boldsymbol{e}^1\otimes\boldsymbol{e}^n,\boldsymbol{e}^2\otimes\boldsymbol{e}^1,\dots,\boldsymbol{e}^n\otimes\boldsymbol{e}^n\}$. Then, any $\mathrm{R}\in\mathcal{L}(\mathfrak{h}\otimes\mathfrak{h})$ may be characterised by its matrix elements

$$\mathrm{R}^{\epsilon'\alpha'}_{\epsilon\,\alpha} \qquad \text{so that} \qquad \mathrm{R}\cdot\boldsymbol{e}^{\epsilon'}\otimes\boldsymbol{e}^{\alpha'} \;=\; \sum_{\{\epsilon,\alpha\}}\mathrm{R}^{\epsilon'\alpha'}_{\epsilon\,\alpha}\boldsymbol{e}^{\epsilon}\otimes\boldsymbol{e}^{\alpha}\;. \tag{2.11}$$

This entails the rule for matrix products

$$(\mathrm{AB})^{\epsilon'\alpha'}_{\epsilon\,\alpha} \;=\; \sum_{\{\tau,\varrho\}}\mathrm{A}^{\tau\,\varrho}_{\epsilon\,\alpha}\,\mathrm{B}^{\epsilon'\alpha'}_{\tau\,\varrho} \tag{2.12}$$

To make the best of this situation in the case of the $L\times M$ grid of vertices as in Fig. 2, with each line i we associate a Hilbert space $\mathfrak{h}_{a_i}=\mathbb{C}^n$ and with each column j a Hilbert space $\mathfrak{h}_j=\mathbb{C}^n$. The spaces $\mathfrak{h}_{a_i}$ or $\mathfrak{h}_j$ are called local. Then $\mathrm{R}^{(i,j)}$ may be embedded into $\mathcal{L}(\mathfrak{h}_{a_i}\otimes\mathfrak{h}_{\mathfrak{q}})$ with $\mathfrak{h}_{\mathfrak{q}}=\otimes_{j=1}^{L}\mathfrak{h}_j$ as $\mathrm{R}^{(i,j)}_{a_i j}$. This notation means that

$$\mathrm{R}^{(i,j)}_{a_i 1} \;=\; \mathrm{R}^{(i,j)}\otimes\underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{L-1\text{ times}} \qquad \text{and} \qquad \mathrm{R}^{(i,j)}_{a_i j} \;=\; \mathrm{P}_{1j}\,\mathrm{R}^{(i,j)}_{a_i 1}\,\mathrm{P}_{1j}\;. \tag{2.13}$$

There, P_{1j} is the permutation operator between spaces $\mathfrak{h}_1$ and $\mathfrak{h}_j$:

$$\mathrm{P}_{1j}(\mathrm{e}_{a_i}\otimes\mathrm{v}_1\otimes\cdots\otimes\mathrm{v}_L) \;=\; \mathrm{e}_{a_i}\otimes\mathrm{v}_j\otimes\mathrm{v}_2\otimes\cdots\otimes\mathrm{v}_{j-1}\otimes\mathrm{v}_1\otimes\mathrm{v}_{j+1}\otimes\cdots\otimes\mathrm{v}_L\;. \tag{2.14}$$

Within the introduced formalism, and under periodic boundary conditions (2.10), we are now in position to evaluate, in a simpler way, the associated normalising factor, *aka* the partition function of the model subject to periodic boundary conditions:

$$\mathcal{Z}_{\mathrm{gen;per}} \;=\; \sum_{C\in\mathcal{E}_{\mathrm{per}}}\prod_{i=1}^{M}\prod_{j=1}^{L}\left\{\left[\mathrm{R}^{(i,j)}\right]^{\epsilon_{i,j}\;\alpha_{i,j}}_{\epsilon_{i,j-1}\alpha_{i-1,j}}\right\}\;. \tag{2.15}$$

Above, the summation runs through all the possible configurations $C=\{\epsilon_{i,j},\alpha_{i,j}\}$ of the grid's edges:

$$\mathcal{E}_{\mathrm{per}} \;=\; \Big\{\alpha_{i,j},\epsilon_{i,j}\in[\![\,1\,;\,n\,]\!] \quad \text{for} \quad i\in[\![\,1\,;\,M\,]\!] \quad \text{and} \quad j\in[\![\,1\,;\,L\,]\!] \quad \text{and} \quad \epsilon_{i,0}=\epsilon_{i,L}\;, \quad \alpha_{0,j}=\alpha_{M,j}\Big\} \tag{2.16}$$

The purely combinatorial expression for $\mathcal{Z}_{\mathrm{gen;per}}$ given in (2.15) may be recast in an elegant way by means of the operator notations we have just established. Indeed, one may reorganise the sums as

$$\begin{aligned}\mathcal{Z}_{\mathrm{gen;per}} \;=\; &\sum_{\{\alpha_{i,j}\}_{i=1,j=1}^{M,L}}\left\{\sum_{\{\epsilon_{1,j}\}_{j=1}^{M}}\left[\mathrm{R}^{(1,1)}\right]^{\epsilon_{1,1}\;\alpha_{1,1}}_{\epsilon_{1,0}\alpha_{0,1}}\left[\mathrm{R}^{(1,2)}\right]^{\epsilon_{1,2}\;\alpha_{1,2}}_{\epsilon_{1,1}\alpha_{0,2}}\cdot\left[\mathrm{R}^{(1,L)}\right]^{\epsilon_{1,L}\;\alpha_{1,L}}_{\epsilon_{1,L-1}\alpha_{0,L}}\right\}\\
&\times\left\{\sum_{\{\epsilon_{2,j}\}_{j=1}^{M}}\left[\mathrm{R}^{(2,1)}\right]^{\epsilon_{2,1}\;\alpha_{2,1}}_{\epsilon_{2,0}\alpha_{1,1}}\left[\mathrm{R}^{(2,2)}\right]^{\epsilon_{2,2}\;\alpha_{2,2}}_{\epsilon_{2,1}\alpha_{1,2}}\cdot\left[\mathrm{R}^{(2,L)}\right]^{\epsilon_{2,L}\;\alpha_{2,L}}_{\epsilon_{2,L-1}\alpha_{1,L}}\right\}\\
&\times\left\{\sum_{\{\epsilon_{M,j}\}_{j=1}^{M}}\left[\mathrm{R}^{(M,1)}\right]^{\epsilon_{M,1}\;\alpha_{M,1}}_{\epsilon_{M,0}\alpha_{M-1,1}}\left[\mathrm{R}^{(M,2)}\right]^{\epsilon_{M,2}\;\alpha_{M,2}}_{\epsilon_{M,1}\alpha_{M-1,2}}\cdot\left[\mathrm{R}^{(M,L)}\right]^{\epsilon_{M,L}\;\alpha_{M,L}}_{\epsilon_{M,L-1}\alpha_{M-1,L}}\right\}\;.\end{aligned} \tag{2.17}$$

There, for fixed i, it is understood that

$$\sum_{\{\epsilon_{i,j}\}_{j=1}^{L}} = \sum_{\epsilon_{i,1}=1}^{n} \cdots \sum_{\epsilon_{i,L}=1}^{n} . \tag{2.18}$$

Now, for fixed $i \in [\![1 ; M]\!]$, each sum may be recast as

$$\sum_{\{\epsilon_{i,j}\}} \Big[\mathrm{R}^{(i,1)}\Big]^{\epsilon_{i,1}\ \alpha_{i,1}}_{\epsilon_{i,0}\alpha_{i-1,1}} \Big[\mathrm{R}^{(i,2)}\Big]^{\epsilon_{i,2}\ \alpha_{i,2}}_{\epsilon_{i,1}\alpha_{i-1,2}} \cdot \Big[\mathrm{R}^{(i,L)}\Big]^{\epsilon_{i,L}\ \alpha_{i,L}}_{\epsilon_{i,L-1}\alpha_{i-1,L}}$$
$$= \sum_{\{\epsilon_{i,L}\}} \Big[\mathrm{R}^{(i,1)}\mathrm{R}^{(i,2)}\cdots\mathrm{R}^{(i,L)}\Big]^{\epsilon_{i,L}\ \alpha_{i,1}\ldots\alpha_{i,L}}_{\epsilon_{i,0}\alpha_{i-1,1}\ldots,\alpha_{i-1,L}} = \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big[\mathrm{T}^{(i)}_{a_i;\mathfrak{q}}\Big]^{\alpha_{i,1}\ldots\alpha_{i,L}}_{\alpha_{i-1,1}\ldots,\alpha_{i-1,L}} . \tag{2.19}$$

Above, we have introduced the monodromy matrix on $\mathfrak{h}_{a_i} \otimes \mathfrak{h}_{\mathfrak{q}}$

$$\mathrm{T}^{(i)}_{a_i;\mathfrak{q}} = \mathrm{R}^{(i,1)}_{a_i\,1} \cdots \mathrm{R}^{(i,L)}_{a_i\,L} , \tag{2.20}$$

and computed its partial trace over $\mathfrak{h}_{a_i}$. The resulting quantity $\mathrm{tr}_{\mathfrak{h}_{a_i}}\Big[\mathrm{T}^{(i)}_{a_i;\mathfrak{q}}\Big]$ is called a transfer matrix and is an operator on $\mathfrak{h}_{\mathfrak{q}}$

This recasts the partition function as

$$\mathcal{Z}_{\mathrm{gen;per}} = \sum_{\{\alpha_{i,j}\}_{i=1,j=1}^{L,M}} \mathrm{tr}_{\mathfrak{h}_{a_1}}\Big[\mathrm{T}^{(1)}_{a_1;\mathfrak{q}}\Big]^{\alpha_{1,1}\ldots\alpha_{1,L}}_{\alpha_{0,1}\ldots,\alpha_{0,L}} \cdot \mathrm{tr}_{\mathfrak{h}_{a_2}}\Big[\mathrm{T}^{(2)}_{a_2;\mathfrak{q}}\Big]^{\alpha_{2,1}\ldots\alpha_{2,L}}_{\alpha_{1,1}\ldots,\alpha_{1,L}} \cdots \mathrm{tr}_{\mathfrak{h}_{a_M}}\Big[\mathrm{T}^{(M)}_{a_M;\mathfrak{q}}\Big]^{\alpha_{M,1}\ldots\alpha_{M,L}}_{\alpha_{M-1,1}\ldots,\alpha_{M-1,L}}$$
$$= \sum_{\{\alpha_{0,1},\ldots,\alpha_{0,M}\}} \Big[\mathrm{tr}_{\mathfrak{h}_{a_1}}\Big[\mathrm{T}^{(1)}_{a_1;\mathfrak{q}}\Big]\cdots\mathrm{tr}_{\mathfrak{h}_{a_M}}\Big[\mathrm{T}^{(M)}_{a_M;\mathfrak{q}}\Big]\Big]^{\alpha_{M,1},\ldots,\alpha_{M,M}}_{\alpha_{0,1},\ldots,\alpha_{0,M}} = \mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\mathrm{tr}_{\mathfrak{h}_{a_1}}\Big[\mathrm{T}^{(1)}_{a_1;\mathfrak{q}}\Big]\cdots\mathrm{tr}_{\mathfrak{h}_{a_M}}\Big[\mathrm{T}^{(M)}_{a_M;\mathfrak{q}}\Big]\Big] . \tag{2.21}$$

So far, the formalism allows one to boil down the summation problem into a linear algebra problem.

Evaluating this normalisation factor, in the thermodynamic limit constitutes the first physically interesting quantities associated with such a model. The associated quantity is called the per-site free energy and is defined by the below limit†

$$\mathfrak{f}_{\mathrm{gen;per}} = -\lim_{M,L\to\infty} \frac{1}{ML} \ln \mathcal{Z}_{\mathrm{gen;per}} . \tag{2.22}$$

Several questions immediately arise:

i) What is the meaning of the limit, *viz.* how do M, L should approach to infinity?

ii) For which class of weights is the limit well-defined?

iii) When defined, is it possible to say more about the limit, for instance how it behaves as a function of the additional parameters that may enter in the expression for the weights, such as the temperature as in (2.3)?

†Is is customary in statistical mechanics to define the per-site free energy with a temperature T-factor appearing in the *rhs* of (2.22). However, since we use the general parametrisation of weights -and not the Boltzmann factor related one as given in (2.3)- we find that dropping it would be more natural in our context.

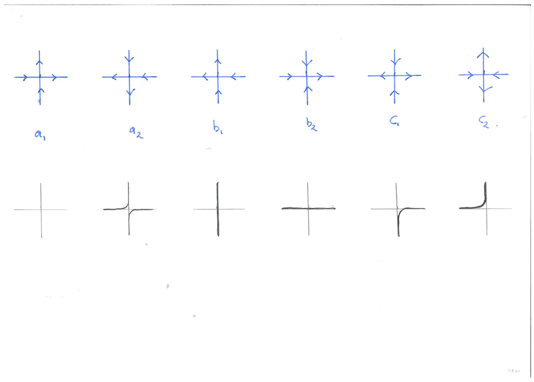

Figure 3: Allowed local configurations in the six-vertex model.

The formalism of rigorous statistical mechanics [6] provides a general background for taking such thermodynamic, *viz.* when the size of the discrete lattice approaches infinity.

One may wonder if alternative ways of computing the limit, for instance as a successive limit

$$-\lim_{M\to\infty}\lim_{L\to\infty}\frac{1}{ML}\ln\mathcal{Z}_{\mathrm{gen;per}} \qquad \text{or} \qquad -\lim_{L\to\infty}\lim_{M\to\infty}\frac{1}{ML}\ln\mathcal{Z}_{\mathrm{gen;per}} \tag{2.23}$$

also exist and whether these limits provide one with the same quantity $\mathfrak{f}_{\mathrm{gen;per}}$. We shall establish such a statement in a later part of these lectures.

However, even when one is in some nice setting allowing one to deal with points *i*) and *ii*), answering question *iii*) turn out to be notoriously hard.

In the following, we will not try to address these questions in full generality, but shall rather focus in answering these -and many other- in the case of a very specific model: the six-vertex model. As we shall see, this model is closely connected to the XXZ spin-$1/2$ Hamiltonian in 1 dimensions mentioned in the introduction. The reason is that this specific model possesses a hidden algebraic structure which allows one to deduce the answer to question *iii*) and many other related ones.

2.2 The six-vertex model

The six-vertex model is a very specific case of a vertex model on an $L\times M$ grid, where each vertex has two possible configurations, *viz.* $n = 2$. As a consequence, these can also be labelled by $\{\pm\}$ or, equivalently, by incoming or out-going arrows, as depicted in Fig. 1. There are, in total, six possible configurations of arrows around a vertex as depicted in Fig. 3. All other configurations are not allowed, *viz.* correspond to a zero weight.

One may denote the weights associated to the allowed configurations as $a_1, a_2, b_1, b_2, c_1, c_2$, *c.f.* Fig. 3. One may solve the model [1], *i.e.* access to explicit expressions for numerous physically interesting observables, for any choices of $a_1, a_2, b_1, b_2, c_1, c_2$. However, for the sake of simplifying the exposition , we shall henceforth limit ourselves to the case

$$a_1 = a_2 = a\,, \qquad b_1 = b_2 = b \quad \text{and} \quad c_1 = c_2 = c\,, \tag{2.24}$$

which already exhibits very rich physical properties.

Let $\mathfrak{h} \simeq \mathbb{C}^2$ be endowed with the basis $\{\boldsymbol{e}^+, \boldsymbol{e}^-\}$. Then, $\mathfrak{h}\otimes\mathfrak{h}$ has basis $\{\boldsymbol{e}^+\otimes\boldsymbol{e}^+, \boldsymbol{e}^+\otimes\boldsymbol{e}^-, \boldsymbol{e}^-\otimes\boldsymbol{e}^+, \boldsymbol{e}^-\otimes\boldsymbol{e}^-\}$

The weights of the six-vertex model on which we focus may be gathered in a 4×4 matrix $\mathtt{R}$:

$$\mathtt{R} = \begin{pmatrix} \mathtt{R}^{++}_{++} & 0 & 0 & 0 \\ 0 & \mathtt{R}^{+-}_{+-} & \mathtt{R}^{-+}_{+-} & 0 \\ 0 & \mathtt{R}^{+-}_{-+} & \mathtt{R}^{-+}_{-+} & 0 \\ 0 & 0 & 0 & \mathtt{R}^{--}_{--} \end{pmatrix} \qquad \text{with} \qquad \begin{cases} \mathtt{R}^{++}_{++} = \mathtt{R}^{--}_{--} = a \\ \mathtt{R}^{+-}_{+-} = \mathtt{R}^{-+}_{-+} = b \\ \mathtt{R}^{-+}_{+-} = \mathtt{R}^{+-}_{-+} = c \end{cases} . \tag{2.25}$$

In fact, one may conveniently parameterise these in terms of three parameters γ, λ and η as

$$\mathtt{R}(\lambda) = \gamma \begin{pmatrix} \sinh(\lambda+\eta) & 0 & 0 & 0 \\ 0 & \sinh(\lambda) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(\lambda) & 0 \\ 0 & 0 & 0 & \sinh(\lambda+\eta) \end{pmatrix} . \tag{2.26}$$

This is the so-called six-vertex $\mathtt{R}$-matrix in polynomial normalisation.

While for generic complex values of γ, λ and η the weights are genuinely complex valued, however there are several possible choices which lead to positive weights.

First of all, the choices $\lambda, \eta, \gamma \in \mathbb{R}^+$ and $\lambda, \eta, \gamma \in \mathbb{R}^-$ yield that the three weights

$$[\mathtt{R}]^{++}_{++} = \gamma \sinh(\lambda+\eta) , \quad [\mathtt{R}]^{+-}_{+-} = \gamma \sinh(\lambda) \quad \text{and} \quad [\mathtt{R}]^{+-}_{-+} = \gamma \sinh(\eta) , \tag{2.27}$$

are all strictly positive. Further, the choice

$$\eta = \mathrm{i}(\pi - \zeta) , \quad \gamma = \frac{-\mathrm{i}r}{\sin(\zeta/2)} , \quad \lambda = \mathrm{i}\frac{\theta\zeta}{\pi} , \tag{2.28}$$

with $\zeta, \theta \in]0\,;\pi[$ and $r \in \mathbb{R}^+$, yields that the three weights

$$[\mathtt{R}]^{++}_{++} = r\frac{\sin[(1-\theta/\pi)\zeta]}{\sin[\zeta/2]} , \quad [\mathtt{R}]^{+-}_{+-} = r\frac{\sin[\theta\zeta/\pi]}{\sin[\zeta/2]} \quad \text{and} \quad [\mathtt{R}]^{+-}_{-+} = 2r\cos[\zeta/2] , \tag{2.29}$$

are also all strictly positive.

The form of the local weight being settled, when passing to the weights for the full $L \times M$ grid, as earlier on, one may also allow the weights to depend in a certain way on the vertex position (i, j) by taking $\mathtt{R}^{(i,j)} = \mathtt{R}(\lambda_i - \xi_j)$ in which $\mathtt{R}(\lambda)$ is the six-vertex R-matrix in polynomial normalisation given above (2.26). Here, in principle, λ_i, ξ_k are some free parameters. However, one should take these of a specific type for all local weights to be real-valued and positive. Namely, either one takes $\gamma, \eta \in \mathbb{R}^+$ and $\lambda_j - \xi_k \in \mathbb{R}^+$, or $\gamma, \eta \in \mathbb{R}^-$ and $\lambda_j - \xi_k \in \mathbb{R}^-$ or

$$\eta = \mathrm{i}(\pi - \zeta) , \quad \gamma = \frac{-\mathrm{i}r}{\sin(\zeta/2)} , \quad \lambda_k = -\mathrm{i}\frac{\zeta}{2} + \mathrm{i}\frac{\theta_k\zeta}{\pi} \quad \text{and} \quad \xi_k = -\mathrm{i}\frac{\zeta}{2} + \mathrm{i}\frac{\sigma_k\zeta}{\pi} , \tag{2.30}$$

where $\zeta \in]0\,;\pi[$, $\lambda_j - \xi_k \in]0\,;\pi[$ and $r \in \mathbb{R}^+$.

By introducing the monodromy matrix on $\mathfrak{h}_{a_i} \otimes \mathfrak{h}_{\mathfrak{q}}$

$$\mathtt{T}_{a_i;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L) = \mathtt{R}_{a_i 1}(\lambda - \xi_1) \cdots \mathtt{R}_{a_i L}(\lambda - \xi_L) \qquad \text{with} \qquad \boldsymbol{\xi}_L = (\xi_1, \dots, \xi_L) , \tag{2.31}$$

and the associated transfer matrix

$$\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L) = \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big[\mathtt{T}_{a_i;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L)\Big] , \tag{2.32}$$

one may identify the partition function $\mathcal{Z}_{6V;\mathrm{per}}$ subordinate to such vertex dependent weights as the below trace

$$\mathcal{Z}_{6V;\mathrm{per}} = \mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\mathtt{t}(\lambda_1 \mid \boldsymbol{\xi}_L) \cdots \mathtt{t}(\lambda_M \mid \boldsymbol{\xi}_L)\Big] . \tag{2.33}$$

We are now going to investigate in greater details the algebraic properties of the various building blocks of the partition function.

2.3 The algebraic structure at the root of the six-vertex model

The main building block of the operators -monodromy and transfer matrices- which provide a simple algebraic expression for $\mathcal{Z}_{6V;\mathrm{per}}$ is the so-called six-vertex R-matrix given in (2.26). One may check that this matrix satisfies an algebraically very nice equation called the Yang-Baxter equation

$$R_{12}(\lambda-\mu)\,R_{13}(\lambda-\nu)\,R_{23}(\mu-\nu) \;=\; R_{23}(\mu-\nu)\,R_{13}(\lambda-\nu)\,R_{12}(\lambda-\mu)\,. \tag{2.34}$$

We shall not establish the validity of this equation here. This can be done through direct calculations based on elementary algebra of trigonometric functions. The equation may also be checked directly as follows. First, one observes that both, *lhs* and *rhs* are hyperbolic polynomials of degree 2, *viz.* is of the form $\sum_{s=-2}^{2} \mathrm{e}^{s\lambda} C_s$ for some coefficients C_s. It is thus enough to check the validity of the equation at 6 points, *e.g.* $\lambda \to \pm\infty$, $\lambda = \mu$, $\lambda = \nu$ and $\lambda = \mu - \eta$, $\lambda = \nu - \eta$.

The Yang-Baxter equation is at the root of the integrability of the 6 vertex model. In fact, when properly generalised - this aspect will not be tackled in the present series of lectures- this equation is at the root on theory of quantum integrable systems. The equation originally appeared in quite different contexts, in the works of McGuire [5], Yang [8] in 1967 and Baxter [2] in 1972. However, it was only in 1979 that it was raised to full glory by Faddeev, Sklyanin and Takhtadjan [3] who showed how to use the algebra generated by this equation for obtaining the exact solution of an integrable model.We shall develop this formalism in this subsection.

First of all, we introduce new objects called Lax matrices

$$\mathrm{L}_{a_i n}(\lambda) \;=\; \gamma \begin{pmatrix} \sinh\big(\lambda + \frac{\eta}{2}[1\,+\,\sigma_n^z]\big) & \sinh(\eta)\cdot\sigma_n^- \\ \sinh(\eta)\cdot\sigma_n^+ & \sinh\big(\lambda + \frac{\eta}{2}[1\,-\,\sigma_n^z]\big) \end{pmatrix}_{[a_i]} \;=\; \mathrm{R}_{a_i n}(\lambda)\,. \tag{2.35}$$

Here $\sigma^z, \sigma^{\pm}$ are the Pauli matrices

$$\sigma^+ \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma^z \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.36}$$

while σ_n^α stand for their embeddings as operators on $\mathfrak{h}_q = \bigotimes_{a=1}^{L} \mathfrak{h}_a$:

$$\sigma_n^\alpha \;=\; \underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{n-1\text{ times}}\otimes\sigma^\alpha\otimes\underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{L-n\text{ times}}\,. \tag{2.37}$$

Further, the index $[a_i]$ in the matrix indicates the space -$\mathfrak{h}_{a_i}$- in respect to which one writes the matrix representation. Note that the entries of this matrix act as non-trivial operators on $\mathfrak{h}_n$. Obviously, by construction, L satisfies the Yang-Baxter equation:

$$R_{ab}(\lambda-\mu)\,\mathrm{L}_{an}(\lambda-\xi_n)\,\mathrm{L}_{bn}(\mu-\xi_n) \;=\; \mathrm{L}_{bn}(\mu-\xi_n)\,\mathrm{L}_{an}(\lambda-\xi_n)\,R_{ab}(\lambda-\mu)\,. \tag{2.38}$$

With these new notation at hand, the monodromy matrix may be written as

$$\mathrm{T}_{a_i;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L) \;=\; \mathrm{L}_{a_i 1}(\lambda-\xi_1)\cdots\mathrm{L}_{a_i L}(\lambda-\xi_L)\,. \tag{2.39}$$

The first impressive result which follows from the algebraic structure at the root of the Yang-Baxter equation is that $\mathrm{T}_{a_i;\mathfrak{q}}$ also satisfies the Yang-Baxter equation.

Proposition 2.1. *It holds*

$$R_{ab}(\lambda-\mu)\,T_{a;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L)\,T_{b;\mathfrak{q}}(\mu \mid \boldsymbol{\xi}_L) \;=\; T_{b;\mathfrak{q}}(\mu \mid \boldsymbol{\xi}_L)\,T_{a;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L)\,R_{ab}(\lambda-\mu)\,. \tag{2.40}$$

Proof —

By using the ultralocal structure of the L matrices, *viz.* that $\mathrm{L}_{an}(\lambda-\xi_n)$ only involves a non trivial action on the spaces $\mathfrak{h}_a$ and $\mathfrak{h}_n$ and thus commutes with all operators acting non-trivially on other spaces that $\mathfrak{h}_a$ and $\mathfrak{h}_n$, one may re-organise the *lhs* of (2.40) in the form

$$\mathrm{R}_{ab}(\lambda-\mu)\,\mathrm{T}_{a;\mathfrak{q}}(\lambda\mid\boldsymbol{\xi}_L)\,\mathrm{T}_{b;\mathfrak{q}}(\mu\mid\boldsymbol{\xi}_L) \;=\; \mathrm{R}_{ab}(\lambda-\mu)\mathrm{L}_{a\,1}(\lambda-\xi_1)\mathrm{L}_{b\,1}(\mu-\xi_1)\mathrm{L}_{a\,2}(\lambda-\xi_2)\mathrm{L}_{b\,2}(\mu-\xi_2) \cdots\mathrm{L}_{a\,L}(\lambda-\xi_L)\mathrm{L}_{b\,L}(\mu-\xi_L)\;. \quad (2.41)$$

Now, one may use the Yang-Baxter equation satisfied by L to successively exchange the "a" and "b" Lax matrices

$$\mathrm{R}_{ab}(\lambda-\mu)\,\mathrm{T}_{a;\mathfrak{q}}(\lambda\mid\boldsymbol{\xi}_L)\,\mathrm{T}_{b;\mathfrak{q}}(\mu\mid\boldsymbol{\xi}_L) \;=\; \mathrm{L}_{b\,1}(\mu-\xi_1)\mathrm{L}_{a\,1}(\lambda-\xi_1)R_{ab}(\lambda-\mu)\mathrm{L}_{a\,2}(\lambda-\xi_2)\mathrm{L}_{b\,2}(\mu-\xi_2)$$
$$\cdots\mathrm{L}_{a\,L}(\lambda-\xi_L)\mathrm{L}_{b\,L}(\mu-\xi_L) \;=\; \mathrm{L}_{b\,1}(\mu-\xi_1)\mathrm{L}_{a\,1}(\lambda-\xi_1)\cdots\mathrm{L}_{b\,j-1}(\mu-\xi_{j-1})\mathrm{L}_{a\,j-1}(\lambda-\xi_{j-1})R_{ab}(\lambda-\mu)$$
$$\times\,\mathrm{L}_{a\,j}(\lambda-\xi_j)\mathrm{L}_{b\,j}(\mu-\xi_j)\cdots\mathrm{L}_{a\,L}(\lambda-\xi_L)\mathrm{L}_{b\,L}(\mu-\xi_L) \;=\; \mathrm{T}_{b;\mathfrak{q}}(\mu\mid\boldsymbol{\xi}_L)\,\mathrm{T}_{a;\mathfrak{q}}(\lambda\mid\boldsymbol{\xi}_L)\,\mathrm{R}_{ab}(\lambda-\mu)\;. \quad (2.42)$$

This entails the claim. ■

While the Yang-Baxter equation for the Lax matrix follows from simple algebra and does appear as a rather complicated re-statement of addition formulae for trigonometric functions, its monodromy matrix variant is by far less trivial. In fact, it allows one to establish several interesting properties, for instance that transfer matrices at different values of the spectral parameter commute:

Lemma 2.2. *Given any $\lambda,\mu\in\mathbb{C}$, $\boldsymbol{\xi}_L\in\mathbb{C}^L$, the transfer matrices defined through* (2.32) *commute:*

$$\Big[\mathtt{t}(\lambda\mid\boldsymbol{\xi}_L)\,,\;\mathtt{t}(\mu\mid\boldsymbol{\xi}_L)\Big] \;=\; 0\;. \quad (2.43)$$

Proof — Observe that

$$\det\Big[\mathrm{R}(\lambda-\mu)\Big] \;=\; \sinh^2(\lambda+\eta)\Big\{\sinh^2(\lambda)-\sinh^2(\eta)\Big\} \;=\; \sinh^3(\lambda+\eta)\sinh(\lambda-\eta)\;. \quad (2.44)$$

Hence, $\mathrm{R}(\lambda-\mu)$ is invertible provided that $\lambda-\mu\notin\{-\eta+\mathrm{i}\pi\mathbb{Z},\eta+\mathrm{i}\pi\mathbb{Z}\}$. Focusing first on spectral parameters satisfying this constraint, one may recast the Yang-Baxter equation in the form

$$\mathrm{T}_{a;\mathfrak{q}}(\lambda\mid\boldsymbol{\xi}_L)\,\mathrm{T}_{b;\mathfrak{q}}(\mu\mid\boldsymbol{\xi}_L) \;=\; \mathrm{R}^{-1}_{ab}(\lambda-\mu)\,\mathrm{T}_{b;\mathfrak{q}}(\mu\mid\boldsymbol{\xi}_L)\,\mathrm{T}_{a;\mathfrak{q}}(\lambda\mid\boldsymbol{\xi}_L)\,\mathrm{R}_{ab}(\lambda-\mu)\;. \quad (2.45)$$

At this stage, one takes the trace over $\mathfrak{h}_a\otimes\mathfrak{h}_b$. On the one hand, it holds

$$\mathrm{tr}_{\mathfrak{h}_a\otimes\mathfrak{h}_b}\Big[\mathrm{T}_{a;\mathfrak{q}}(\lambda\mid\boldsymbol{\xi}_L)\,\mathrm{T}_{b;\mathfrak{q}}(\mu\mid\boldsymbol{\xi}_L)\Big] \;=\; \mathtt{t}(\lambda\mid\boldsymbol{\xi}_L)\cdot\mathtt{t}(\mu\mid\boldsymbol{\xi}_L) \quad (2.46)$$

since the tensor product trace factorises for pure-tensor product matrices. On the other hand, by using the cyclicity of the trace, one has

$$\mathrm{tr}_{\mathfrak{h}_a\otimes\mathfrak{h}_b}\Big[\mathrm{R}^{-1}_{ab}(\lambda-\mu)\,\mathrm{T}_{b;\mathfrak{q}}(\mu\mid\boldsymbol{\xi}_L)\,\mathrm{T}_{a;\mathfrak{q}}(\lambda\mid\boldsymbol{\xi}_L)\,\mathrm{R}_{ab}(\lambda-\mu)\Big]$$
$$=\;\mathrm{tr}_{\mathfrak{h}_a\otimes\mathfrak{h}_b}\Big[\,\mathrm{T}_{b;\mathfrak{q}}(\mu\mid\boldsymbol{\xi}_L)\,\mathrm{T}_{a;\mathfrak{q}}(\lambda\mid\boldsymbol{\xi}_L)\,\mathrm{R}_{ab}(\lambda-\mu)\,\mathrm{R}^{-1}_{ab}(\lambda-\mu)\Big]$$
$$=\;\mathrm{tr}_{\mathfrak{h}_a\otimes\mathfrak{h}_b}\Big[\,\mathrm{T}_{b;\mathfrak{q}}(\mu\mid\boldsymbol{\xi}_L)\,\mathrm{T}_{a;\mathfrak{q}}(\lambda\mid\boldsymbol{\xi}_L)\Big] \;=\; \mathtt{t}(\mu\mid\boldsymbol{\xi}_L)\cdot\mathtt{t}(\lambda\mid\boldsymbol{\xi}_L)\;, \quad (2.47)$$

what entails that $\mathtt{t}(\mu\mid\boldsymbol{\xi}_L)$ and $\mathtt{t}(\lambda\mid\boldsymbol{\xi}_L)$ commute provided that $\lambda-\mu\neq\pm\eta \mod \mathrm{i}\pi$. Since $\mathtt{t}(\mu\mid\boldsymbol{\xi}_L)$ is an operator valued hyperbolic polynomial -*viz.* a polynomials in $\mathrm{e}^{-\lambda},\mathrm{e}^{\lambda}$- taken that the relation

$$\Big[\mathtt{t}(\lambda\mid\boldsymbol{\xi}_L)\,,\;\mathtt{t}(\mu\mid\boldsymbol{\xi}_L)\Big] \;=\; 0\;, \quad (2.48)$$

holds for almost all values of $\lambda - \mu$, it actually holds for all values. This entails the claim. ■

The commutativity of the transfer matrices has a strong impact on the computationability of the partition function. To discuss this matter, we shall first list several properties of the R-matrix which can be obtained from elementary algebra. These will then allow us to deduce less trivial properties enjoyed by the transfer matrix. The R-matrix (2.32) enjoys the Hermitian conjugation

$$[\mathrm{R}_{0k}(\lambda)]^{\dagger_k} \;=\; -\frac{\gamma^*}{\gamma}\mathrm{R}_{0k}^{t_0}(-\lambda^*) \quad \text{if} \quad \eta \in \mathrm{i}\mathbb{R} \qquad \text{and} \qquad [\mathrm{R}_{0k}(\lambda)]^{\dagger_k} \;=\; \frac{\gamma^*}{\gamma}\mathrm{R}_{0k}^{t_0}(\lambda^*) \quad \text{if} \quad \eta \in \mathbb{R}\,. \tag{2.49}$$

in which t_a, resp. $\dagger_a$, stands for the partial transpose, resp. Hermitian conjugation, on the space a. Furthermore, it satisfies the crossing symmetry properties

$$\sigma_1^x\,\mathrm{R}_{12}^{t_1}(\lambda + \mathrm{i}\pi - \eta)\,\sigma_1^x \;=\; R_{21}(-\lambda) \qquad \text{and} \qquad \sigma_1^y\,\mathrm{R}_{12}^{t_1}(\lambda - \eta)\,\sigma_1^y \;=\; -R_{21}(-\lambda) \tag{2.50}$$

Finally, it also satisfies the unitarity property

$$\mathrm{R}(\lambda)\mathrm{R}(-\lambda) \;=\; -\sinh(\lambda+\eta)\sinh(\lambda-\eta)\cdot \mathrm{I}_4 \tag{2.51}$$

where I_n stands for the identity matrix in n-dimensions.

The above equations imply, in the case of specific inhomogeneities, a simple behaviour of the monodromy matrix under hermitian conjugation on the quantum space. To state the result, it is convenient to represent the monodromy matrix $\mathrm{T}_{a;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L)$ as a 2×2 matrix on the space $\mathfrak{h}_a$ whose entries are operators on $\mathfrak{h}_{\mathfrak{q}}$:

$$\mathrm{T}_{a;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L) \;=\; \begin{pmatrix} \mathrm{A}(\lambda \mid \boldsymbol{\xi}_L) & \mathrm{B}(\lambda \mid \boldsymbol{\xi}_L) \\ \mathrm{C}(\lambda \mid \boldsymbol{\xi}_L) & \mathrm{D}(\lambda \mid \boldsymbol{\xi}_L) \end{pmatrix}_{[a]}\,. \tag{2.52}$$

From now on, as long as this will not lead to confusion, we shall drop the dependence on the auxiliary parameters in the operators A, B, C and D, *viz.* write

$$\mathrm{T}_{a;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L) \;=\; \begin{pmatrix} \mathrm{A}(\lambda) & \mathrm{B}(\lambda) \\ \mathrm{C}(\lambda) & \mathrm{D}(\lambda) \end{pmatrix}_{[a]}\,. \tag{2.53}$$

Lemma 2.3. *Assume that the parameters are given as*

$$\eta \;=\; \mathrm{i}(\pi - \zeta)\,, \quad \gamma \;=\; \frac{-\mathrm{i}r}{\sin(\zeta/2)}\,, \quad \lambda \;=\; -\mathrm{i}\frac{\zeta}{2} + \mathrm{i}\frac{\theta\zeta}{\pi}\,, \quad \xi_k \;=\; -\mathrm{i}\frac{\zeta}{2} + \check{\xi}_k\,, \tag{2.54}$$

with $r, \check{\xi}_k \in \mathbb{R}$ *and* $\theta, \zeta \in]0\,;\pi[$.

Then, upon denoting $\dagger_{\mathfrak{q}}$ *the Hermitian conjugation on* $\mathfrak{h}_{\mathfrak{q}}$*, one has*

$$\begin{pmatrix} [A(\lambda)]^{\dagger_{\mathfrak{q}}} & [B(\lambda)]^{\dagger_{\mathfrak{q}}} \\ [C(\lambda)]^{\dagger_{\mathfrak{q}}} & [D(\lambda)]^{\dagger_{\mathfrak{q}}} \end{pmatrix}_{[a]} \;=\; \begin{pmatrix} D(\check{\lambda}) & C(\check{\lambda}) \\ B(\check{\lambda}) & A(\check{\lambda}) \end{pmatrix}_{[a]}\,. \tag{2.55}$$

where $\check{\lambda} \;=\; -\mathrm{i}\frac{\zeta}{2} + \mathrm{i}\frac{(\pi-\theta)\zeta}{\pi}$.

Similarly, let

$$\eta, \gamma, \lambda \in \mathbb{R}^+ \qquad \textit{while} \qquad \xi_k \;=\; \frac{\eta}{2} + \mathrm{i}\check{\xi}_k\,, \tag{2.56}$$

then it holds

$$\begin{pmatrix} [A(\lambda)]^{\dagger_{\mathfrak{q}}} & [B(\lambda)]^{\dagger_{\mathfrak{q}}} \\ [C(\lambda)]^{\dagger_{\mathfrak{q}}} & [D(\lambda)]^{\dagger_{\mathfrak{q}}} \end{pmatrix}_{[a]} \;=\; (-1)^L \begin{pmatrix} D(-\lambda^*) & -C(-\lambda^*) \\ B(-\lambda^*) & A(-\lambda^*) \end{pmatrix}_{[a]}\,. \tag{2.57}$$

Proof —

If $\eta \in \mathrm{i}\mathbb{R}$, then using that one computes a partial Hermitian conjugate and that operators with different space labels commute and further applying the *lhs* formula of of (2.49) and subsequently the crossing relation (2.50) produces the chain of equalities

$$\Big[\mathtt{T}_{a;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L)\Big]^{\dagger_{\mathfrak{q}}} = [\mathrm{R}_{a1}(\lambda-\xi_1)]^{\dagger_1}\cdots[\mathrm{R}_{aL}(\lambda-\xi_L)]^{\dagger_L}$$
$$= \Big(-\frac{\gamma^*}{\gamma}\Big)^L \mathrm{R}_{a1}^{t_a}(\xi_1^*-\lambda^*)\cdots\mathrm{R}_{aL}^{t_a}(\xi_L^*-\lambda^*)$$
$$= \Big(-\frac{\gamma^*}{\gamma}\Big)^L \sigma_a^x\,\mathrm{R}_{a1}(\lambda^*-\xi_1^*+\mathrm{i}\pi-\eta)\cdots\mathrm{R}_{aL}(\lambda^*-\xi_L^*+\mathrm{i}\pi-\eta)\,\sigma_a^x\,. \quad (2.58)$$

Now, one observes that for the choice of parameters (2.54), it holds $-\gamma^*/\gamma = 1$, while

$$\lambda^*-\xi_k^*+\mathrm{i}\pi-\eta = \check{\lambda}-\xi_k \quad (2.59)$$

what yields

$$\Big[\mathtt{T}_{a;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L)\Big]^{\dagger_{\mathfrak{q}}} = \sigma_a^x\,\mathtt{T}_{a;\mathfrak{q}}(\check{\lambda} \mid \boldsymbol{\xi}_L)\,\sigma_a^x\,. \quad (2.60)$$

This entails the first part of the claim.

Now going to the real valued case (2.56), it holds

$$\Big[\mathtt{T}_{a;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L)\Big]^{\dagger_{\mathfrak{q}}} = = \Big(\frac{\gamma^*}{\gamma}\Big)^L \mathrm{R}_{a1}^{t_a}(\lambda^*-\xi_1^*)\cdots\mathrm{R}_{aL}^{t_a}(\lambda^*-\xi_L^*)$$
$$= \Big(-\frac{\gamma^*}{\gamma}\Big)^L \sigma_a^y\,\mathrm{R}_{a1}(\xi_1^*-\lambda^*-\eta)\cdots\mathrm{R}_{aL}(\xi_L^*-\lambda^*-\eta)\,\sigma_a^y\,. \quad (2.61)$$

Since $\dfrac{\gamma^*}{\gamma} = 1$ and $\xi_a^*-\eta = -\xi_a^*$, one infers that

$$\Big[\mathtt{T}_{a;\mathfrak{q}}(\lambda \mid \boldsymbol{\xi}_L)\Big]^{\dagger_{\mathfrak{q}}} = (-1)^L\,\sigma_a^y\,\mathtt{T}_{a;\mathfrak{q}}(-\lambda^* \mid \boldsymbol{\xi}_L)\,\sigma_a^y\,, \quad (2.62)$$

hence leading to the second part of the claim. ■ ■

As a consequence, for the choice (2.54), it holds

$$[\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)]^{\dagger_{\mathfrak{q}}} = [\mathtt{A}(\lambda)+\mathtt{D}(\lambda)]^{\dagger_{\mathfrak{q}}} = \mathtt{A}(\check{\lambda})+\mathtt{D}(\check{\lambda}) = \mathtt{t}(\check{\lambda} \mid \boldsymbol{\xi}_L)\,, \quad (2.63)$$

while, for the choice (2.56),

$$[\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)]^{\dagger_{\mathfrak{q}}} = \mathtt{t}(-\lambda^* \mid \boldsymbol{\xi}_L)\,, \quad (2.64)$$

Thus, for such a choice of parameters, since the transfer matrices commute for different values of their spectral parameters, $\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)$ commutes with its hermitian adjoint. It is thus a normal operator. As such it is diagonalisable and admits an Eigenbasis $\boldsymbol{\Phi}_k$, $k = 0,\dots,2^L-1$ with associated Eigenvalues $\Lambda_k(\lambda \mid \boldsymbol{\xi}_L)$. Thus, taking the trace over $\mathfrak{h}_{\mathfrak{q}}$ in (2.33) in respect to this Eigenbasis allows one to re-express the periodic boundary condition partition function in terms of a spectral sum

$$\mathcal{Z}_{6V;\mathrm{per}} = \sum_{k=0}^{2^L-1}\prod_{a=1}^{M}\Lambda_k(\lambda_k \mid \boldsymbol{\xi}_L)\,. \quad (2.65)$$

Here ξ_k, η are as in (2.54) or (2.56) while

$$\lambda_k = -\mathrm{i}\frac{\zeta}{2} + \mathrm{i}\frac{\theta_k \zeta}{\pi} \quad \text{if} \quad \eta \in \mathrm{i}\mathbb{R} \quad \textit{and} \quad \lambda_k = \frac{\eta}{2} + \theta_k \quad \text{if} \quad \eta \in \mathbb{R} \,. \tag{2.66}$$

The representation (2.65) thus shows that the problem of computing the partition function boils down, in the six-vertex integrable setting, to diagonalising the transfer matrix $\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)$. We shall discuss this question, in great detail, at a later stage of the lectures.

3 The XXZ spin-1/2 chain

3.1 From the six-vertex model to the XXZ spin-1/2 chain

There is a striking relationship between observables related with the six-vertex model and those pertaining to the XXZ spin-1/2 Hamiltonian. This connection was first observed by McCoy and Wu [4] in 1968 and late generalised to the 8 vertex/XYZ chain setting by Sutherland [7] in 1970. In fact, this connection is part of a much general picture which holds for numerous two dimensional models of statistical mechanics/ one-dimensional quantum Hamiltonians.

We first observe that since $\left\{\mathtt{t}(\lambda)\right\}_{\lambda\in\mathbb{C}}$ constitutes a commutative subalgebra of the space of operators on $\mathfrak{h}_{\mathfrak{q}}$ we may expand its elements around a certain points to obtain a commutative algebra of operators, aka a set of conserved quantities. It turns out that, for the homogeneous model $\xi_k = \xi$, such a expansion point is conveniently chosen to be $\lambda = \xi$.

Proposition 3.1. *Let $\xi_k = \xi \in \mathbb{C}$. Then, it holds*

$$\mathtt{t}(\xi \mid \boldsymbol{\xi}_L) = (\gamma \sinh \eta)^L \mathcal{T}_L \qquad \textit{with} \qquad \mathcal{T}_L = \mathcal{P}_{12} \cdots \mathcal{P}_{1L} \tag{3.1}$$

being the backward translation operator by one site

$$\mathcal{T}_L \cdot v_1 \otimes \cdots \otimes v_L = v_2 \otimes \cdots \otimes v_{L-1} \otimes v_1 \,. \tag{3.2}$$

Moreover, it holds

$$2J \sinh(\eta) \cdot \mathtt{t}^{-1}(\xi \mid \boldsymbol{\xi}_L) \cdot \partial_\lambda \mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)_{|\lambda=\xi} = 2J \sinh(\eta) \partial_\lambda \ln \mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)_{|\lambda=\xi} = \mathbb{H}_0 \tag{3.3}$$

where

$$\mathbb{H}_h = J \sum_{a=1}^{L} \left\{ \sigma_a^x \sigma_{a+1}^x + \sigma_a^y \sigma_{a+1}^y + \cosh(\eta)(\sigma_a^z \sigma_{a+1}^z + \mathrm{id}) \right\} - \frac{h}{2} \sum_{a=1}^{L} \sigma_a^z \tag{3.4}$$

*is the celebreated XXZ spin-*1/2 *Hamiltonian in an external magnetic field.*

Proof —

One observes that the R-matrix becomes proportional to the permutation operator at the origin

$$R_{ab}(0) = \gamma \sinh(\eta) \mathrm{P}_{ab} \qquad \text{with} \qquad \mathtt{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.5}$$

$\mathtt{P}_{ab}$ is such that, for $a < b$,

$$\mathtt{P}_{ab}\mathtt{v}_1 \otimes \cdots \otimes \mathtt{v}_a \otimes \cdots \otimes \mathtt{v}_b \otimes \cdots \otimes \mathtt{v}_L \; = \; \mathtt{v}_1 \otimes \cdots \otimes \mathtt{v}_b \otimes \cdots \otimes \mathtt{v}_a \otimes \cdots \otimes \mathtt{v}_L \; . \tag{3.6}$$

These operators satisfy the algebra

$$\mathtt{P}_{ab}\mathtt{P}_{cb} \; = \; \mathtt{P}_{ca}\mathtt{P}_{ab} \; . \tag{3.7}$$

Owing to (2.35), one has $L_{a_i n}(0) \; = \; \gamma \sinh(\eta)\mathtt{P}_{a_i n}$. Hence, going back to the definition of the transfer matrix,

$$\begin{aligned} \mathtt{t}(\xi \mid \boldsymbol{\xi}_L) \; &= \; \big(\gamma \sinh(\eta)\big)^L \cdot \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big[\mathtt{P}_{a_i 1} \cdots \mathtt{P}_{a_i L}\Big] \\ &= \; \big(\gamma \sinh(\eta)\big)^L \cdot \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big[\mathtt{P}_{12} \cdots \mathtt{P}_{1L}\mathtt{P}_{a_i 1}\Big] \; = \; \big(\gamma \sinh(\eta)\big)^L \cdot \mathtt{T}_L \cdot \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big[\mathtt{P}_{a_i 1}\Big] \end{aligned} \tag{3.8}$$

Here, one observes that if $\mathtt{E}^{\epsilon\epsilon'}$ stands for the elementary 2×2 matrix having zero entries everywhere except on the intersection of line ϵ and column ϵ', then

$$\mathtt{P}_{ab} \; = \; \sum_{\epsilon,\epsilon'=1}^{2} \mathtt{E}_a^{\epsilon\epsilon'} \mathtt{E}_b^{\epsilon'\epsilon} \; . \tag{3.9}$$

Hence,

$$\mathrm{tr}_{\mathfrak{h}_a}\Big[\mathtt{P}_{ab}\Big] \; = \; \sum_{\epsilon,\epsilon'=1}^{2} \underbrace{\mathrm{tr}\Big[\mathtt{E}^{\epsilon\epsilon'}\Big]}_{=\delta_{\epsilon,\epsilon'}} \mathtt{E}_b^{\epsilon'\epsilon} \; = \; \sum_{\epsilon=1}^{2} \mathtt{E}_b^{\epsilon\epsilon} \; = \; \mathrm{id} \; . \tag{3.10}$$

This entails the claim relative to the first equation of the Proposition. The fact that $\mathtt{T}_L$ is a backward translation operator by one site can be checked directly

$$\begin{aligned} & \mathtt{T}_L \mathtt{v}_1 \otimes \cdots \otimes \mathtt{v}_L \; = \; \mathtt{P}_{12} \cdots \mathtt{P}_{1L-1}\mathtt{v}_L \otimes \mathtt{v}_2 \otimes \cdots \otimes \mathtt{v}_{L-1} \otimes \mathtt{v}_1 \\ & \quad = \; \mathtt{P}_{12} \cdots \mathtt{P}_{1L-2}\mathtt{v}_{L-1} \otimes \mathtt{v}_2 \otimes \cdots \otimes \mathtt{v}_{L-2} \otimes \mathtt{v}_L \otimes \mathtt{v}_1 \\ & \qquad = \; \cdots \; = \; \mathtt{v}_2 \otimes \cdots \otimes \mathtt{v}_{L-1} \otimes \mathtt{v}_1 \; . \end{aligned} \tag{3.11}$$

Further, one computes

$$\begin{aligned} \partial_\lambda \mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)_{|\lambda=\xi} \; &= \; \big(\gamma \sinh(\eta)\big)^{L-1} \sum_{k=1}^{L} \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big[\mathtt{P}_{a_i 1} \cdots \mathtt{P}_{a_i k-1}\partial_\lambda \mathtt{L}_{a_i k}(0)\mathtt{P}_{a_i k+1} \cdots \mathtt{P}_{a_i L}\Big] \\ &= \; \big(\gamma \sinh(\eta)\big)^{L-1} \sum_{k=1}^{L} \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big[\mathtt{P}_{a_i 1} \cdots \mathtt{P}_{a_i k-1}\mathtt{P}_{a_i k}\mathtt{P}_{a_i k}\partial_\lambda \mathtt{L}_{a_i k}(0)\mathtt{P}_{a_i k+1} \cdots \mathtt{P}_{a_i L}\Big] \\ &= \; \big(\gamma \sinh(\eta)\big)^{L-1} \sum_{k=1}^{L} \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big[\mathtt{P}_{k-1,k}\partial_\lambda \mathtt{L}_{k-1k}(0) \cdot \mathtt{P}_{a_i 1} \cdots \mathtt{P}_{a_i k-1}\mathtt{P}_{a_i k}\mathtt{P}_{a_i k+1} \cdots \mathtt{P}_{a_i L}\Big] \\ &= \; \big(\gamma \sinh(\eta)\big)^{L-1} \sum_{k=1}^{L} \mathtt{P}_{k-1k}\partial_\lambda \mathtt{L}_{k-1k}(0) \cdot \mathtt{T}_L \; . \end{aligned} \tag{3.12}$$

Now, a direct calculation yields

$$\mathtt{P}_{ab}\partial_\lambda \mathtt{L}_{ab}(0) \;=\; \gamma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[ab]} \times \begin{pmatrix} \cosh(\eta) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cosh(\eta) \end{pmatrix}_{[ab]} \;=\; \gamma \begin{pmatrix} \cosh(\eta) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cosh(\eta) \end{pmatrix}_{[ab]} \tag{3.13}$$

It may then be checked by means of direct calculations that

$$\frac{1}{2}\Big\{\sigma^x \otimes \sigma^x \;+\; \sigma^y \otimes \sigma^y + \cosh(\eta)(\sigma^z \otimes \sigma^z + \mathrm{id})\Big\} \;=\; \begin{pmatrix} \cosh(\eta) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cosh(\eta) \end{pmatrix} . \tag{3.14}$$

Hence,

$$\begin{aligned} \partial_\lambda \mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)_{|\lambda=\xi} \;&=\; \frac{1}{2\sinh(\eta)} \sum_{a=1}^{L} \Big\{\sigma_a^x \otimes \sigma_{a+1}^x \;+\; \sigma_a^y \otimes \sigma_{a+1}^y + \cosh(\eta)(\sigma_a^z \otimes \sigma_{a+1}^z + \mathrm{id})\Big\} \mathtt{t}(\xi \mid \boldsymbol{\xi}_L) \\ &=\; \frac{\mathtt{t}(\xi \mid \boldsymbol{\xi}_L)}{2\sinh(\eta)} \sum_{a=1}^{L} \Big\{\sigma_a^x \otimes \sigma_{a+1}^x \;+\; \sigma_a^y \otimes \sigma_{a+1}^y + \cosh(\eta)(\sigma_a^z \otimes \sigma_{a+1}^z + \mathrm{id})\Big\} \end{aligned} \tag{3.15}$$

since the sum, taken as whole, is translation invariant. This entails the claim. ■

We have just established that the XXZ spin-1/2 Hamiltonian in a vanishing external magnetic field commutes with -and in fact is begot by- the family of transfer matrices $\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)$ arising in the study of the six-vertex model! This means that it is enough to construct the Eigenvectors and Eigenvalues of $\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)$ so as to immediately access these of the XXZ chain! Taken that the transfer matrix is built out of elements $\mathtt{A}(\lambda), \mathtt{B}(\lambda), \mathtt{C}(\lambda), \mathtt{D}(\lambda)$ which enjoy nice algebraic exchange relation which follows from the Yang-Baxter equation for the monodromy matrix, *c.f* Proposition 2.1, it turns out to be much easier to consider the diagonalisation problem directly on the level of $\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)$ instead of the XXZ Hamiltonian. This will be explained in the next section. However, first, we shall establish that, in fact, diagonalising $\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)$ allows one to diagonalise $\mathtt{H}_h$ as well.

Lemma 3.2. *Let*

$$S^z \;=\; \sum_{a=1}^{L} \sigma_a^z \tag{3.16}$$

be the total spin operator on $\mathfrak{h}_{\mathfrak{q}}$*. Then, it holds*

$$\Big[\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L), S^z\Big] \;=\; 0\,. \tag{3.17}$$

Proof—

One starts with the observation that

$$\Big[\sigma_a^z + \sigma_b^z, \mathtt{L}_{ab}(\lambda)\Big] \;=\; 0 \tag{3.18}$$

which can be inferred from a direct calculation. Thus,

$$\begin{aligned}
\Big[\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L), \mathtt{S}^z\Big] &= \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big\{\Big[\mathrm{L}_{a_i 1}(\lambda-\xi_1)\cdots \mathrm{L}_{a_i L}(\lambda-\xi_L), \mathtt{S}^z\Big]\Big\} \\
&= \sum_{k=1}^{L} \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big\{\mathrm{L}_{a_i 1}(\lambda-\xi_1)\cdots \mathrm{L}_{a_i k-1}(\lambda-\xi_{k-1})\cdot\Big[\mathrm{L}_{a_i k}(\lambda-\xi_k), \mathtt{S}^z\Big]\cdot \mathrm{L}_{a_i k+1}(\lambda-\xi_{k+1})\cdots \mathrm{L}_{a_i L}(\lambda-\xi_L)\Big\} \\
&= \sum_{k=1}^{L} \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big\{\mathrm{L}_{a_i 1}(\lambda-\xi_1)\cdots \mathrm{L}_{a_i k-1}(\lambda-\xi_{k-1})\cdot\Big[\mathrm{L}_{a_i k}(\lambda-\xi_k), \sigma^z_k\Big]\cdot \mathrm{L}_{a_i k+1}(\lambda-\xi_{k+1})\cdots \mathrm{L}_{a_i L}(\lambda-\xi_L)\Big\} \\
&= -\sum_{k=1}^{L} \mathrm{tr}_{\mathfrak{h}_{a_i}}\Big\{\mathrm{L}_{a_i 1}(\lambda-\xi_1)\cdots \mathrm{L}_{a_i k-1}(\lambda-\xi_{k-1})\cdot\Big[\mathrm{L}_{a_i k}(\lambda-\xi_k), \sigma^z_{a_i}\Big]\cdot \mathrm{L}_{a_i k+1}(\lambda-\xi_{k+1})\cdots \mathrm{L}_{a_i L}(\lambda-\xi_L)\Big\} \\
&= -\mathrm{tr}_{\mathfrak{h}_{a_i}}\Big\{\Big[\mathrm{L}_{a_i 1}(\lambda-\xi_1)\cdots \mathrm{L}_{a_i L}(\lambda-\xi_L), \sigma^z_{a_i}\Big]\Big\} \;=\; 0\,, \qquad (3.19)
\end{aligned}$$

since the trace of a commutator vanishes. ■

Observe that the Hilbert space $\mathfrak{h}_{\mathfrak{q}}$ of the XXZ chain decomposes into the direct sum of Eigenspaces of the total spin operator

$$\mathfrak{h}_{\mathfrak{q}} \;=\; \bigoplus_{N=0}^{L} \mathfrak{h}_{\mathfrak{q}}^{(N)} \qquad \text{with} \qquad \mathfrak{h}_{\mathfrak{q}}^{(N)} \;=\; \Big\{\boldsymbol{v}\in\mathfrak{h}_{\mathfrak{q}} \;:\; \mathtt{S}^z\cdot\boldsymbol{v} = (L-2N)\cdot\boldsymbol{v}\Big\}\,. \qquad (3.20)$$

The commutation property established in the previous lemma thus entails that $\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)$ is block diagonal in respect to the above Hilbert space direct sum decomposition. As a consequence, any Eigenvector of $\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)$ belongs to a subsector $\mathfrak{h}_{\mathfrak{q}}^{(N)}$ for some N. Hence, if $\Lambda_k(\lambda \mid \boldsymbol{\xi}_L)$ is the Eigenvalue of $\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)$ associated with the Eigenvector $\boldsymbol{\Phi}_k \in \mathfrak{h}_{\mathfrak{q}}^{(N)}$,

$$\mathtt{S}^z\cdot\boldsymbol{\Phi}_k \;=\; (L-2N)\boldsymbol{\Phi}_k \qquad (3.21)$$

and thus, by virtue of Proposition 3.1,

$$\mathtt{H}_h\cdot\boldsymbol{\Phi}_k \;=\; \Big\{2J\sinh(\eta)\partial_\lambda \ln\Lambda_k(\lambda \mid \boldsymbol{\xi}_L)_{|\lambda=\xi} \;-\; \frac{h}{2}(L-2N)\Big\}\cdot\boldsymbol{\Phi}_k\,, \qquad (3.22)$$

where $\xi_a = \xi$ for any $a \in [\![\,1\,;\,L\,]\!]$. Note that there is no problem in differentiating the Eigenvalues in respect to λ since $\mathtt{t}(\lambda \mid \boldsymbol{\xi}_L)$ being a commuting family of hyperbolic polynomials, the Eigenvalues are hyperbolic polynomials as well.

References

[1] R.J. Baxter, *"Generalised Ferroelectric model on a square lattice"*, Stud. Appl. Math. **L** (1971), 51–69.

[2] ———, *"Partition function of the eight vertex lattice model."*, Ann. Phys. **70** (1972), 193–228.

[3] L.D. Faddeev, E.K. Sklyanin, and L.A. Takhtadzhan, *"Quantum inverse problem method I."*, Teor. Math. Phys. **40** (1979), 688–706.

[4] B.M. McCoy and T.T. Wu, *"Hydrogen-Bonded Crystals and the Anisotropic Heisenberg Chain."*, Il Nuovo Cimento B **56** (1968), 311–315.

[5] J.B. McGuire, *"Study of exactly soluble one-dimensional N-body problem."*, J. Math. Phys. **5** (1964), 622–636.

[6] D. Ruelle, *"Statistical mechanics: rigorous results"*, W.A. Benjamin, Inc., 1969.

[7] B. Sutherland, *"Two-dimensional hydrogen bonded crystals without the ice rule."*, Phys. Rev. Lett. **19** (1970), 3183–3186.

[8] C.N. Yang, *"Some exact results for the many-body problem in one dimension with repulsive delta-function interaction."*, Phys. Rev. Lett. **19** (1967), 1312–1315.