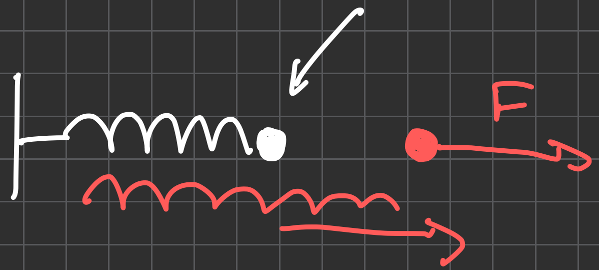
M - a manifold

TM

$L : TM \longrightarrow \mathbb{R}$

$$L(q,\dot q) = \frac{m}{2} g_{ij}\, \dot q^i \dot q^j - V(q)$$

metric on M

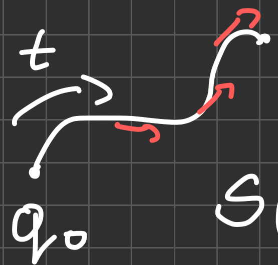


$M = \mathbb{R}^{3n}$

$M = SO(3)$

M – Minkowski space ...

Włodzimierz Tulczyjew



$$S(q(t)) = \int_{t_0}^{t_1} L(\dot q(t))\,dt$$

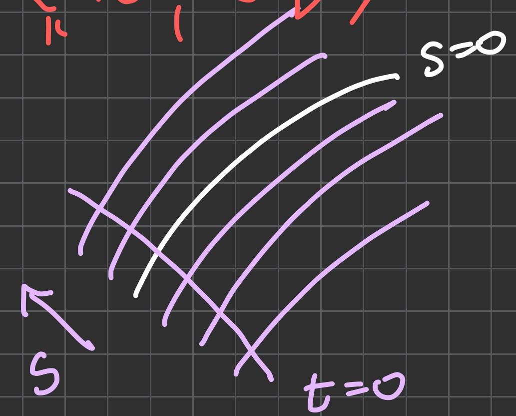
$\dot q : \mathbb{R} \longrightarrow TM$

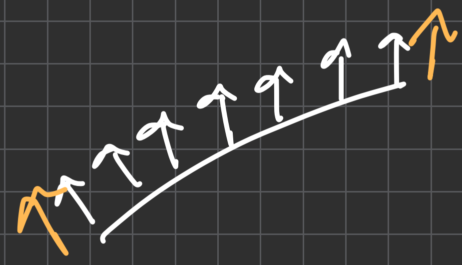
$M \ (x^i)$

$$\delta S(\delta q) = \frac{d}{ds}\Big|_{s=0} \int_{t_0}^{t_1} L\left(x^i(t,s), \dot{x}^i(t,s)\right) dt =$$

$\langle dL, \delta(\dot{x})\rangle$

$\langle$ "dS", $\delta q \rangle$



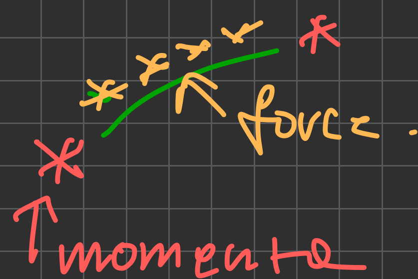


$$\int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} \delta x^i(t,0) + \frac{\partial L}{\partial \dot{x}^i} \delta \dot{x}^i(t,0) \right) dt =$$

$\delta(\dot{x})$

$(\delta x)^{\cdot}$

$\delta S \longleftrightarrow$

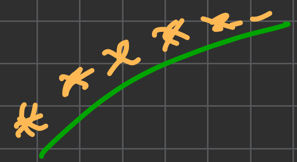


$\chi : \mathbb{R}^2 \longrightarrow M$

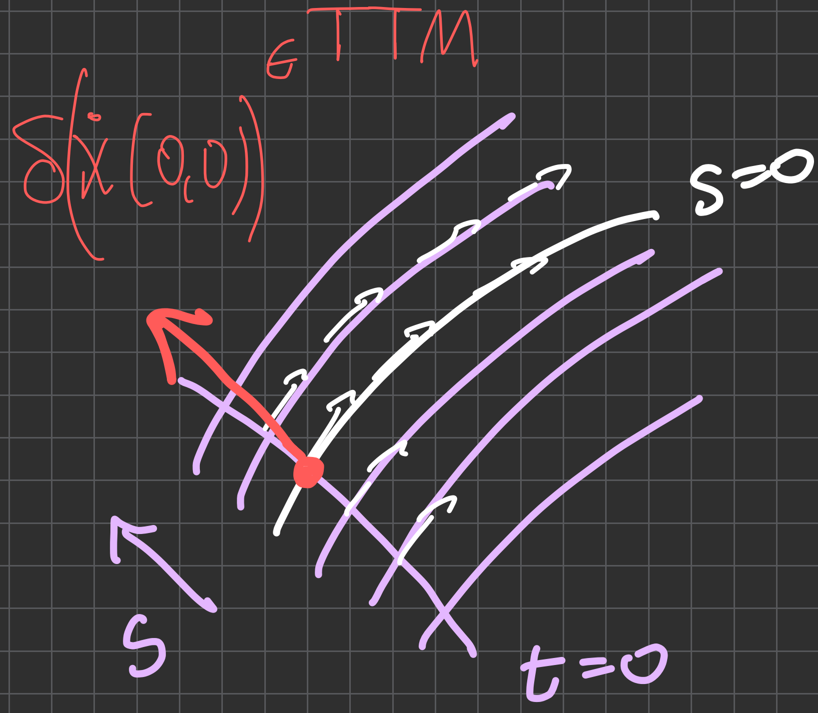
$q(t) = \chi(t,0)$

$\chi^i = x^i \circ \chi$

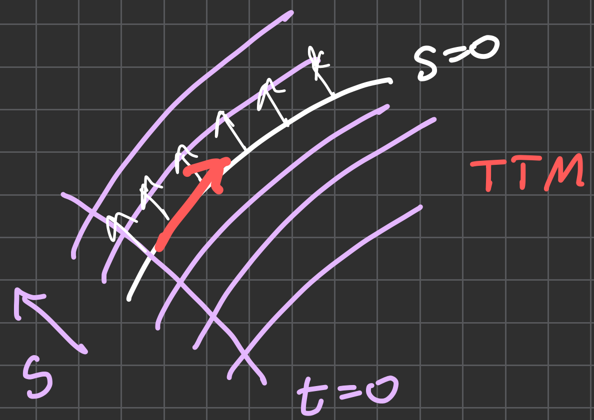
$$= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial x^i} \delta x^i(t,0) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i} \delta x^i(t,0) \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i} \right) \delta x^i(t,0) \right\} dt =$$



$$= \int_{t_0}^{t_1} \overbrace{\left\{ \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i} \right) \right\}}^{f_i} \delta x^i(t,0)\, dt + \underbrace{\frac{\partial L}{\partial \dot{x}^i}}_{\text{momento} \ \in T^*M} \delta x^i(t,0) \Big|_{t_0}^{t_1}$$

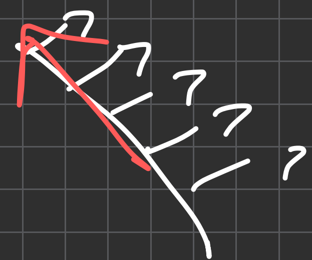


$\dot{x}$ $\qquad$ $(\delta x)^{\cdot}$



$$K_M : TTM \longrightarrow TTM$$

$$(x^i, \dot{x}^j, \delta x^k, \delta\dot{x}^m) \longmapsto (x^i, \delta x^j, \dot{x}^k, \delta\dot{x}^m)$$



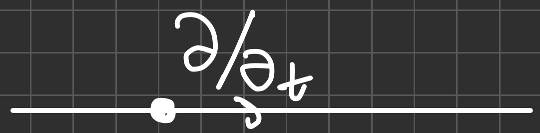
$$= \int_{t_0}^{t_1} \Big\{ \overbrace{\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)}^{f_i} \Big\} \delta x^i(t,0)\, dt + \underbrace{\frac{\partial L}{\partial \dot{x}^i}}_{p_i} \delta x^i(t,0) \Big|_{t}^{t_1}$$

momento $\rightarrow$ $\in T^*M$

$$\langle \delta S, \delta \varphi \rangle = \int_{t_0}^{t_1} \langle dL, \delta \dot{x}(t,0) \rangle\, dt =$$

$$= \int \left(\langle f, \delta x \rangle\, dt + \frac{d}{dt} \langle p_i, \delta x(t,0) \rangle \right)$$

T^*M $p(t)$ $\quad$ TM $\cup$



$$\langle dL, \delta \dot{x}(t,0) \rangle = \langle f, \delta x \rangle + \frac{d}{dt} \langle p_i, \delta x(t,0) \rangle$$

$$\langle dL, \delta \dot{x}(t,0) \rangle = \langle\langle \underbrace{\dot{p}}_{TT^*M}, \underbrace{(\delta x)^{\cdot}}_{TTM} \rangle\rangle$$

$$
\begin{tikzcd}
TTM \arrow[d, "\tau_{TM}"] \arrow[rrr, bend left, "\kappa_M"] & (q, \dot q, \delta q, \delta \dot q) \arrow[d] & & TTM \arrow[d, "T\tau_M"] & (q, \dot q, \delta q, \delta \dot q) \arrow[d] \\
TM & (q, \dot q) & & TM & (q, \delta q)
\end{tikzcd}
$$

isomorphism of vector bundles

$\tau_M : TM \to M$

$\pi_M : T^*M \to M$

$$
\begin{tikzcd}
T^*TM \arrow[d, "\pi_{TM}"] & TT^*M \arrow[l, "\alpha_M"'] \arrow[d, "T\pi_M"] \\
TM & TM
\end{tikzcd}
$$

α_M is the dual of κ_M with respect to two v.b. structures in TTM

$\langle dL, \delta \dot q \rangle = \langle \dot p, (\delta q)^{\cdot} \rangle$

$\delta \dot q \xrightarrow{\kappa_M} (\delta q)^{\cdot}$

$dL \overset{\alpha_M}{\longleftrightarrow} \dot p$

$\dot p$, $p(t) \in T^*M$

$$D_L = \alpha_M^{-1}(dL(TM)) \subset TT^*M$$

A first order differential equation on curves in T^*M ← positions and momenta.

Examples 1 Mechanical Lagrangian
2 Free relativistic particle

$TTM \quad (q, \dot q, \delta q, \delta \dot q)$

$T^*TM \quad (q, \dot q, \xi, \eta)$

$$\xi_i \delta q^i + \eta_i \delta \dot q^i$$

$$(p\delta q)^{\cdot} = \dot p\, \delta q + p\, \delta \dot q$$

TTM

$TT^*M \quad (q, p, \dot q, \dot p)$

$$p_i = \eta_i \quad \xi_i = \dot p_i$$

$$p_i = \frac{\partial L}{\partial \dot q^i}$$

$$\alpha_M : (q, p, \dot q, \dot p) \longmapsto (q, \dot q, \dot p, p)$$

$$\alpha_m : (q, p, \dot q, \dot p) \longmapsto (q, \dot q, \dot p, p)$$

$$L(q,\dot q) = \frac{m}{2} g_{ij} \dot q^i \dot q^j - V(q)$$

$$\Big(q, \dot q, \underbrace{\frac{m}{2} \frac{\partial g_{ij}}{\partial q^k} \dot q^i \dot q^j - \frac{\partial V}{\partial q^k}}_{\dot p_k}, \underbrace{m g_{ij} \dot q^i}_{p_j}\Big)$$

$$\boxed{\begin{aligned} p_j &= m g_{ij} \dot q^i \\ \dot p_k &= \frac{m}{2} \frac{\partial g_{ij} \dot q^i \dot q^j}{\partial q^k} - \frac{\partial V}{\partial q^k} \end{aligned}}$$

$$\dot q = \cdots$$

$$\dot p = \cdots$$

$$g^{ij} g_{ik} = \delta^i_k$$

$$\dot q^i = \frac{1}{m} g^{ij} p_j$$

$$\dot p_k = \frac{\cancel{m}}{2} \frac{\partial g_{ij}}{\partial q^k} \frac{1}{\cancel{m}} g^{il} p_l \frac{1}{m} g^{jn} p_n - \frac{\partial V}{\partial q^k}$$

$$= -\frac{1}{2m} \frac{\partial g^{il}}{\partial q^k} p_i p_l - \frac{\partial V}{\partial q^k}$$

$$\dot q^i = \frac{\partial H}{\partial p_i}$$

$$\dot p_k = -\frac{\partial H}{\partial q^k}$$

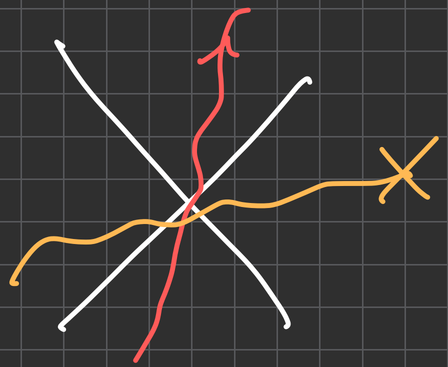
Hamilton equations for $H(q,p) = \frac{1}{2m} g^{ij} p_i p_j + V$

Free relativistic particle

$TM = M \times V$

(M, η) a Minkowski space (affine)

$T^*M = M \times V^*$



$$L : TM \supset M \times V_+ \longrightarrow \mathbb{R}$$

$\eta(v,v) > 0$

$$L(x,v) = m \overbrace{\sqrt{\eta(v,v)}}^{\|v\|}$$

$+ - - -$

$$TT^*M = M \times V^* \times V \times V^* \xrightarrow{\alpha_M} T^*TM = M \times V \times V^* \times V^*$$

$$(x, \underset{\uparrow}{p}, \dot{x}, \underset{0}{\dot{p}}) \qquad (x, \dot{x}, \dot{p}, p)$$

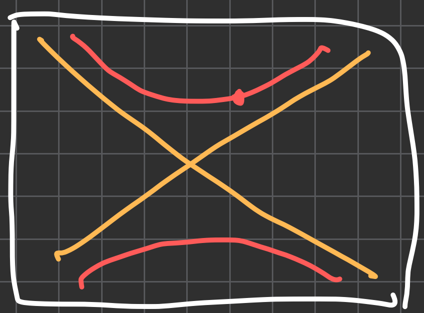
$\dot{x} = r\,\bar{\eta}(p, \cdot)$ $\quad \|p\|^2 = m$

$\dot{p} = 0$

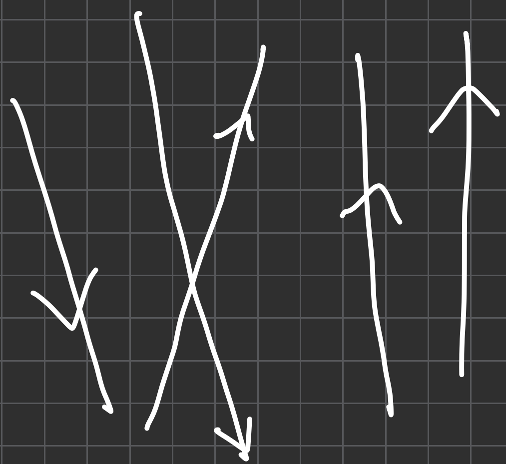
$$dL(x, \dot{x}) = \left(x, \dot{x}, 0, m \frac{2\eta(\dot{x}, \cdot)}{2\sqrt{\eta(\dot{x}, \dot{x})}}\right)$$

$$p = m \cdot \eta\left(\frac{\dot{x}}{\|\dot{x}\|}, \cdot\right)$$

$$\|p\|^2 = m^2$$



V^*



$$T^*T^*M \xleftarrow{\beta_M} TT^*M \xrightarrow{\alpha_M} T^*TM \supset \mathcal{L}(TM)$$

$$(\omega_{T^*M}) \qquad (d_T\omega_M) \qquad (\omega_{TM})$$

symplectomorphism

Lagrangian

$$\begin{array}{c} T^*M \\ \pi_M \downarrow \\ M \end{array} \qquad \begin{array}{ccc} & TT^*M & \\ \tau_{T^*M} \swarrow & & \searrow T\pi_M \\ T^*M & & TM \end{array}$$

$$\alpha_M^* \omega_{TM} = d_T \omega_M$$

$$(T^*M, \omega_M)$$

(P, ω) $d\omega = 0$ ω - nondegenerate

$$TP \ni V \xmapsto{\text{isomorphism}} \omega(V, \cdot) \in T^*P$$

Liouville one form on T^*M θ_M

$q \; p \quad \dot{q} \; \dot{p}$

$? \, dq^i + \dot{s} \, dp_i$

$$\langle \theta_M, \underset{\substack{\cap \\ TT^*M}}{V} \rangle = \langle \tau_{T^*M}(V), T\pi_M(V) \rangle \qquad q \, \underline{p} \qquad q \, \underline{\dot{q}}$$

$$= \dot{q}^i p_i$$

$\theta_M = p_i \, dq^i \qquad d\theta_M = dp_i \wedge dq^i$

(T^*M, ω_M) $\omega_M = dp_i \wedge dq^i$

$f : M \to M$

$$TT^*M \xrightarrow{\alpha_M} T^*TM$$

$d_T\omega_M$

$(q, \dot q, \xi, \eta)$

$d\xi_i \wedge dq^i + d\eta_i \wedge d\dot q^i$

$d_T f : TM \to \mathbb{R}$ $d_T f = df$

$$d_T(\omega_M = dp_i \wedge dq^i)$$

$$d_T\omega_M = d\dot p_i \wedge dq^i + dp_i \wedge d\dot q^i$$

TT^*M $(q, p, \dot q, \dot p)$

$$(-dq^i \wedge dp_i)^\cdot = -d\dot q^i \wedge dp_i - dq^i \wedge d\dot p_i$$

(P, ω) $\quad P = T^*M$

$$T \quad TP \xrightarrow{\tilde{\omega}} T^*P$$
$$v \longmapsto \omega(v, \cdot)$$

$$\begin{tikzcd}
TTP \arrow[r, "T\tilde{\omega}"] & TT^*P \arrow[d, "\alpha_P"] \\
TTP \arrow[u, "\kappa_M"] \arrow[r, "d_T\omega"'] \arrow[r, bend left=60] & T^*TP
\end{tikzcd}$$

Special submanifolds of a symplectic manifold

ALGEBRA $(f_1\, f_2\, f_3, \dots, f_k ; f_{k+1}\, f_{k+2} \cdots f_{2k})$

$(f_1, \dots, f_n)$ $\quad \langle \varphi^i, f_j \rangle = \delta^i_j$

$(\varphi^1, \dots, \varphi^n)$

$n \geqslant 2k$

$\omega = \varphi^1 \wedge \varphi_{k+1} + \varphi^2 \wedge \varphi_{k+2} + \dots + \varphi^k \wedge \varphi_{2k}$

$\underline{\underline{n = 2k}}$

isotropic $W \subset W^{\S}$ $\leqslant k$

coisotropic $W \supset W^{\S}$ $\geqslant k$

lagrangian $W = W^{\S}$ k

symplectic $\omega|_W$ nondegenerate

$W \subset V$

$W^{\S} = \{ v \in V : \forall w \in W \ \ \omega(v,w) = 0 \}$

$\dim W^{\S} = \underset{\substack{\shortparallel \\ n}}{2k} - \dim W$

$N \subset P$ ↙ symplectic vector

$\underbrace{T_q N}_{T} \subset T_q P$

$$f : M \longrightarrow \mathbb{R}$$

$$P = T^* M \supset df(M)$$

Lagrangian